

# CONSTANT MEAN CURVATURE TRINOIDS

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ABSTRACT. This paper constructs a family of constant mean curvature immersions of the thrice-punctured Riemann sphere into  $\mathbb{R}^3$  with asymptotically Delaunay ends via Loop group methods.

## INTRODUCTION

A *trinoid* is a conformal immersion of the thrice-punctured Riemann sphere into  $\mathbb{R}^3$  with constant mean curvature (CMC) with asymptotically Delaunay ends.

CMC trinoids were first constructed in [6]. The family of Alexandrov-embedded trinoids was classified in [5].

In this paper, a family of trinoids is constructed via the Dorfmeister-Pedit-Wu (DPW) [2] construction. This family is three-dimensional, parametrized by the asymptotic necksizes of the ends, and has four connected components, according as the ends are embedded unduloids or immersed nodoids.

Via the DPW construction, every CMC immersion can be obtained by first solving a linear meromorphic ODE

$$d\Phi_\lambda = \Phi_\lambda \xi_\lambda, \quad \Phi_\lambda(z_0) = \Phi_\lambda^0.$$

in 2-by-2 matrices which depend on loop parameter  $\lambda \in \mathbb{S}^1$ . A loop group factorization is then applied to  $\Phi = FB$  to produce the  $SU_2$  frame  $F$  for an associate family of CMC immersions.

The first step in the DPW construction of CMC surfaces is to write down a suitable family of potentials  $\xi_\lambda$  for the ODE. To produce asymptotically Delaunay ends, a natural choice is a potential which is gauge equivalent to a linear superposition of three potentials for Delaunay ends

$$\xi_\lambda = \sum \frac{A_k(\lambda)}{z - z_k} dz.$$

At each end, such a potential  $\xi_\lambda$  has a simple pole whose residue  $A_k(\lambda)$  encodes the asymptotic necksize. That the ends are asymptotic to half

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Delaunay surfaces follows from the fact that at each end the potential is a perturbation of a potential which produces a Delaunay surface.

Since the solution  $\Phi_\lambda$  to the ODE has monodromy around the ends, it is necessary to simultaneously close the ends by choosing a suitable initial condition  $\Phi_\lambda^0$ . The ends are closed when the monodromy representation of  $\Phi_\lambda$  is unitary. In the case of three ends, the necessary and sufficient condition for unitarizing the monodromy representation is the triangle inequalities on the 2-sphere

$$\begin{aligned}\nu_1 + \nu_2 + \nu_3 &\leq 1 \\ \nu_i &\leq \nu_j + \nu_k\end{aligned}$$

where  $\nu_1, \nu_2, \nu_3$  are the necksizes of the three ends, which can be read off from the residues of  $\xi_\lambda$ . (In the case of  $n > 3$  ends, the spherical  $n$ -gon inequalities on the necksizes are a necessary condition.) The  $\Phi_\lambda^0$  which unitarizes the monodromy pointwise in  $\lambda$  is then “glued” holomorphically in  $\lambda$  (theorem 4.9) to produce an initial condition for which the ends of the CMC immersion are closed.

The tools developed here will be useful for constructing further examples of CMC surfaces, including  $n$ -noids in  $\mathbb{R}^3$ ,  $n$ -noids in  $H^3$  and  $\mathbb{S}^3$ , and  $n$ -noids dressed by Bäckland transformations.

The images in this paper were generated with **CMCLab**, a numerical implementation of the DPW algorithm developed by the author using algorithms for loop group factorizations explicated by I. McIntosh. The software, documentation, and a gallery of CMC surface images produced by the software is available at the Center for Geometry, Analysis, Numerics and Graphics (GANG) website, [www.gang.umass.edu](http://www.gang.umass.edu).

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### Outline of the paper.

Section 1 (Preliminary) explicates the DPW construction and provides background theory relating to monodromy, closing conditions and gauge equivalence.

Section 2 (Delaunay immersions) gives the basic background concerning Delaunay surfaces, their dressings and their asymptotic growth rates.

Section 3 (Perturbations of Delaunay immersions) proves the asymptotics theorem 3.4, showing that the CMC immersion arising from a perturbation of a Delaunay DPW potential has an asymptotically Delaunay end.

Section 4 (Unitarization of loop group monodromy representations) proves the gluing theorem 4.9, that under the assumption of the pointwise unitarizability of a monodromy representation on  $\mathbb{S}_\lambda^1$ , there exist an  $r$ -dressing which conjugates the monodromy representation to an  $r$ -unitary representation.

In section 5 (Constructing trinoids) a family of trinoid potentials  $\mathcal{T}$  is given (definition 5.1). Theorem 5.7 gives a necessary and sufficient condition for simultaneous unitarizability of a monodromy representation on the thrice-punctured sphere in terms of the eigenvalues. Theorem 5.12 shows that the monodromy representation for  $\xi \in \mathcal{T}$  is unitarizable pointwise on  $\mathbb{S}_\lambda^1$ . Theorems 5.13 and 5.14 draw on these results to construct a family of trinoids parametrized by the three end weights.

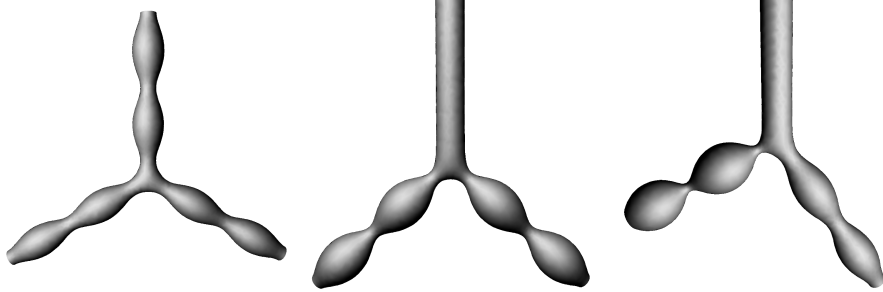


FIGURE 1. Equilateral, isosceles and scalene CMC trinoids with unduloid ends. Their respective necksizes are  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ,  $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$  and  $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ . These examples are static in the sense that the sum of their necksizes is the maximum of 1.

## 1. PRELIMINARY

### 1.1. The DPW construction.

**Notation 1.1.** The following notation is used for circles, disks and annuli in the domain  $\mathbb{P}_\lambda^1$  of the loop parameter  $\lambda$ . Let  $r \in (0, 1]$ .

$$\mathcal{C}_r = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$$

$$\mathcal{D}_r = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$$

$$\mathcal{D}'_r = \{\lambda \in \mathbb{C} \mid |\lambda| > r\} \cup \{\infty\}$$

$$\mathcal{A}_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}.$$

**Notation 1.2.** For a map  $X : A_r \rightarrow M_{k \times k}(\mathbb{C})$ , the star operator is defined as

$$(1.1) \quad X^*(\lambda) = \overline{X(\bar{\lambda}^{-1})}^t.$$

**Notation 1.3.** The following groups are defined for  $G$  either  $\mathrm{GL}_n(\mathbb{C})$  or  $\mathrm{SL}_n(\mathbb{C})$ .

Let  $\mathcal{T}G$  denote the group of upper triangular elements of  $G$  and  $\mathcal{T}^{\mathbb{R}}G$  the subgroup of  $\mathcal{T}G$  whose diagonal elements are in  $\mathbb{R}^+$ . For  $r \in (0, 1]$ ,

- $\Lambda_r G$  (“loops”) is the group of analytic maps  $C_r \rightarrow G$ .
- $\Lambda_1^* G$  (“unitary loops”) is the subgroup of  $\Lambda_1 G$  of loops  $X$  which satisfy the reality condition

$$(1.2) \quad \widehat{X} = \widehat{X}^{*-1}.$$

For  $r \in (0, 1)$ ,  $\Lambda_r^* G$  (“ $r$ -unitary loops”) is the subgroup of  $\Lambda_r G$  of loops  $X$  such that  $X$  is the boundary of an analytic map  $\widehat{X} : \mathcal{A}_r \rightarrow G$  satisfying the reality condition.

- $\Lambda_r^+ G$  is the subgroup of  $\Lambda_r G$  of loops  $X$  such that  $X$  is the boundary of an analytic map  $\widehat{X} : \mathcal{D}_r \rightarrow G$  satisfying  $\widehat{X}(0) \in \mathcal{T}G$ .

- $\Lambda_r^{+, \mathbb{R}} G$  (“positive loops”) is the subgroup of  $\Lambda_r^+ G$  of loops  $X$  such that  $X$  is the boundary of an analytic map  $\widehat{X} : \mathcal{D}_r \rightarrow G$  satisfying  $\widehat{X}(0) \in \mathcal{T}^{\mathbb{R}}G$ .

- $\Lambda_{\uparrow 1}^{+, \mathbb{R}} \mathrm{M}_{2 \times 2}(\mathbb{C})$  is the set of analytic maps  $X : \mathbb{S}^1 \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  such that  $X$  is the boundary of an analytic map  $\widehat{X} : \mathcal{D}_1 \rightarrow \mathrm{GL}_2(\mathbb{C})$  satisfying  $\widehat{X}(0) \in \mathcal{T}^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ .

The  $r$ -Iwasawa factorization [9, 8] is as follows.

**Theorem** (Iwasawa factorization theorem). *Let  $r \in (0, 1]$ . Take  $G$  to be either  $\mathrm{GL}_2(\mathbb{C})$  or  $\mathrm{SL}_2(\mathbb{C})$ . Then any  $X \in \Lambda_r G$  can be factored uniquely*

$$X = X_u X_+$$

with  $X_u \in \Lambda_r^* G$  and  $X_+ \in \Lambda_r^{+, \mathbb{R}} G$ . The induced map

$$\Lambda_r G \rightarrow \Lambda_r^* G \times \Lambda_r^{+, \mathbb{R}} G$$

is an analytic diffeomorphism .

The projections of the  $r$ -Iwasawa factorization of  $X$  to the first and second factors are respectively denoted by  $\mathrm{Uni}_r[X]$  and  $\mathrm{Pos}_r[X]$ . For loops  $F \in \Lambda_r G$  and  $C \in \Lambda_r G$ , the  $r$ -dressing action of  $C$  on  $F$  is  $\mathrm{Uni}_r[CF]$ .

**Notation 1.4.**  $\Lambda_r^{-1} \mathrm{gl}_2(\mathbb{C})$  and  $\Lambda_r^{-1} \mathrm{sl}_2(\mathbb{C})$  are respectively the sets of holomorphic  $\mathrm{gl}_2(\mathbb{C})$ - and  $\mathrm{sl}_2(\mathbb{C})$ -valued functions on  $\mathcal{D}_r$  which extend meromorphically to  $\lambda = 0$  and whose expansion in  $\lambda$  at  $\lambda = 0$  is of the form

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \mathrm{O}(\lambda^0).$$

For a Riemann surface  $\Sigma$  and complex vector space  $V$ ,  $\Omega_\Sigma^1(V)$  denotes the holomorphic  $V$ -valued 1-forms on  $\Sigma$ .

For  $X \in \Lambda_r G$ , the notation  $X'$  means differentiation with respect to  $\theta$ , where  $\lambda = e^{i\theta}$ . We have  $(X')^* = (X^*)'$ .

For  $X \in \mathfrak{gl}_2(\mathbb{C})$ ,  $\text{tracefree}(X) = X - \frac{1}{2}(\text{tr } X)I$ .

The DPW construction [2] is as follows.

**Theorem (DPW).** *Let  $\Sigma$  be a Riemann surface and  $\tilde{\Sigma}$  its universal cover. Let  $r \in (0, 1]$ . Let  $\xi \in \Omega_\Sigma^1(\Lambda_r^{-1}\mathfrak{gl}_2(\mathbb{C}))$ . Let  $z_0 \in \tilde{\Sigma}$  and let  $\Phi_0 \in \Lambda_r \text{GL}_2(\mathbb{C})$ . Let  $\Phi : \tilde{\Sigma} \rightarrow \Lambda_r \text{GL}_2(\mathbb{C})$  be the solution to the initial value problem*

$$(1.3) \quad d\Phi = \Phi\xi; \quad \Phi(z_0) = \Phi_0.$$

*This initial value problem is denoted by the triple  $(\xi, z_0, \Phi_0)$ .*

*Let*

$$\Phi = FB$$

*be the  $r$ -Iwasawa factorization of  $\Phi$ . Then  $F$  extends to a map  $F : \tilde{\Sigma} \rightarrow \Lambda_r^* \text{GL}_2(\mathbb{C})$  and  $F|_{\mathbb{S}^1}$  takes values in  $U_2$ .  $F$  is called the extended frame.*

*Let  $\text{Sym}_\lambda[\cdot]$  be defined on maps  $F : \tilde{\Sigma} \rightarrow \Lambda_1^* \text{GL}_2(\mathbb{C})$  by*

$$(1.4) \quad \text{Sym}_\lambda[F] = -2H^{-1} \text{tracefree}(F'F^{-1}).$$

*For each  $\lambda \in \mathbb{S}^1$ , the map  $\text{Sym}_\lambda[F]$  is a conformal constant mean curvature immersion  $\Sigma \rightarrow \text{su}_2 \equiv \mathbb{R}^3$  with mean curvature  $H$ . The family  $\text{Sym}_\lambda[F]$  over  $\lambda \in \mathbb{S}^1$  is an associate family of CMC immersions.*

**Remark 1.1.** The Hopf differential of  $f_\lambda$  is  $-2H^{-1}\alpha\beta\lambda^{-1}$  and its metric is  $4H^{-2}R^2\alpha \otimes \bar{\alpha}$ , where  $R = B_{11}/B_{22}$  and  $(B_{ij}) = B|_{\lambda=0}$ .

## 1.2. Monodromy and closing conditions.

**Lemma 1.5.** *Let  $0 < r_1 < r_2 \leq 1$ , and suppose that  $\Phi_1 \in \Lambda_{r_1} \text{GL}_2(\mathbb{C})$ ,  $\Phi_2 \in \Lambda_{r_2} \text{GL}_2(\mathbb{C})$  are the boundary of an analytic map  $\Phi : \{r_1 < |\lambda| < r_2\}$ . Let  $\Phi_j = F_j B_j$  be the  $r_j$ -Iwasawa factorizations of  $\Phi_j$ ,  $j = 1, 2$ . Let  $F_j$  be the extension of  $F_j$  to  $\mathcal{A}_{r_j}$  and  $B_j$  be the extension of  $B_j$  to  $\mathcal{D}_{r_j}$ . Then  $F_2$  extends analytically to  $\mathcal{A}_{r_1}$  and is equal to  $F_1$  there, and  $B_1$  extends analytically to  $\mathcal{D}_{r_2}$  and is equal to  $B_2$  there.*

*Proof.* Since  $\Phi$  and  $B_2$  are analytic on  $\{r_1 < |\lambda| < r_2\}$ ,  $F_2 = \Phi_2 B_2^{-1}$  extends analytically to  $\mathcal{A}_{r_1}$ . Since  $\Phi$  and  $F_1$  are analytic on  $\{r_1 < |\lambda| < r_2\}$ ,  $B_1 = F_1^{-1}\Phi_1$  extends analytically to  $\mathcal{D}_{r_2}$ . Hence  $\Phi = F_1 B_1 = F_2 B_2$  is an  $r$ -Iwasawa factorization for any  $r \in [r_1, r_2]$ , so by the uniqueness of  $r$ -Iwasawa factorization,  $F_1 = F_2$  and  $B_1 = B_2$ .  $\square$

**Notation 1.6.** Let  $\Sigma$  be a Riemann surface,  $\tilde{\Sigma} \rightarrow \Sigma$  its universal cover, and  $\Gamma$  the group of deck transformations for this cover. Let  $r \in (0, 1]$ , let  $\xi \in \Omega_{\Sigma}^1(\Lambda_r^{-1}\mathfrak{gl}_2(\mathbb{C}))$  and let  $\Phi : \tilde{\Sigma} \rightarrow \Lambda_r \mathrm{GL}_2(\mathbb{C})$  be a solution to the ODE  $d\Phi = \Phi\xi$ . The *monodromy representation* of  $\Phi$  is the map  $M_{\Phi} : \Gamma \rightarrow \Lambda_r \mathrm{GL}_2(\mathbb{C})$  defined by  $M_{\Phi}(\tau) = (\tau^*\Phi)\Phi^{-1}$ .

In the case  $\Sigma = \Sigma_0 \setminus \{p_1, \dots, p_n\}$  for a closed Riemann surface  $\Sigma_0$ , we define the ‘‘monodromy of  $\Phi$  at  $p_k$ ’’ as  $M_{\Phi}(\tau)$ , where  $\tau \in \Gamma$  is defined as follows: let  $U$  be an annular neighborhood of  $p_k$ ,  $\gamma : [0, 1] \rightarrow U$  a closed curve with winding number 1 around  $p_k$ , and  $\tau \in \Gamma$  the deck transformation satisfying  $\tau(\gamma(0)) = \gamma(1)$ .

**Lemma 1.7.** *Let  $\Sigma$ ,  $\tilde{\Sigma}$ ,  $\Gamma$ ,  $r$ ,  $\xi$ ,  $\Phi$  and  $M$  be as in notation 1.6, and suppose that*

$$(1.5) \quad M \in \Lambda_r^* \mathrm{GL}_2(\mathbb{C}).$$

Let  $\lambda_0 \in \mathbb{S}^1$ , and let  $f_{\lambda} = \mathrm{Sym}_{\lambda}[\mathrm{Uni}_r[\Phi]]$ . Let  $\tau \in \Gamma$ . Then  $(\tau^*F)F^{-1}$  is  $z$ -independent,  $(\tau^*B)B^{-1} = \mathrm{I}$ , and the following are equivalent:

$$(1.6) \quad \begin{aligned} &M(\tau, \lambda_0) \text{ is a multiple of } \mathrm{I} \\ &\mathrm{tracefree}(M'(\tau, \lambda_0)) = 0 \end{aligned}$$

and

$$(1.7) \quad \tau^*f_{\lambda_0} = f_{\lambda_0},$$

where  $f = \mathrm{Sym}[F]$ . In the case  $M \in \Lambda_r^* \mathrm{SL}_2(\mathbb{C})$ , conditions (1.6) are equivalent to

$$(1.8) \quad M(\tau, \lambda_0) = \pm \mathrm{I}, \quad M'(\tau, \lambda_0) = 0.$$

*Proof.* Let  $\Phi = FB$  be the  $r$ -Iwasawa factorization of  $\Phi$ . Then

$$(1.9) \quad (\tau^*F^{-1})(M(\tau))F = (\tau^*B)B^{-1}.$$

holds on  $\mathcal{C}_r$ . Since  $M \in \Lambda_r^* \mathrm{GL}_2(\mathbb{C})$ , the left hand side of equation (1.9) is in  $\Lambda_r^* \mathrm{GL}_2(\mathbb{C})$  while the right hand side is in  $\Lambda_r^{+, \mathbb{R}} \mathrm{GL}_2(\mathbb{C})$ . The uniqueness of the  $r$ -Iwasawa factorization implies that each side of the equation is  $\mathrm{I}$ , so  $M(\tau) = (\tau^*F)F^{-1}$  and  $(\tau^*B)B^{-1} = \mathrm{I}$  on  $\mathcal{C}_r$ . Under the assumption (1.5),  $M(\tau)$  is in  $\Lambda_r^* \mathrm{GL}_2(\mathbb{C})$ , so it extends analytically to  $\mathcal{A}_r$ .

We have

$$(1.10) \quad \tau^*f = M(\tau)fM(\tau)^{-1} - 2H^{-1} \mathrm{tracefree}(M(\tau)'M(\tau)^{-1}).$$

Assuming equation (1.6),  $M(\tau, \lambda_0) = \pm \mathrm{I}$  and  $M'(\tau, \lambda_0) = 0$ , so the formula (1.10) evaluated at  $\lambda_0$  yields  $\tau^*f_{\lambda_0} = f_{\lambda_0}$ .

Conversely, note that for fixed  $\lambda \in \mathbb{S}^1$ , the action on  $f_{\lambda}$  defined by the right hand side of equation (1.10) is an isometry of  $\mathfrak{su}_2$ . If

equation (1.7) holds, then this isometry fixes  $f_{\lambda_0}$  pointwise, so either  $f_{\lambda_0}$  lies in two-dimensional subspace of  $\mathfrak{su}_2$  or the isometry is the identity. Equations (1.6) follow.  $\square$

The following lemma shows that condition 1.6 can be replaced by an analogous condition on the eigenvalues of  $M_\Phi$ .

**Lemma 1.8.** *Let  $\gamma$  be an open segment of  $\mathbb{S}^1$ ,  $M : \gamma \rightarrow \mathbb{U}_2$  an analytic map,  $\rho_1, \rho_2$  the eigenvalues of  $M$ , and  $\lambda_0 \in \gamma$ . Then the conditions (1.6) are equivalent to*

$$(1.11) \quad \rho_1(\lambda_0) = \rho_2(\lambda_0), \quad \rho'_1(\lambda_0) = \rho'_2(\lambda_0).$$

*In the case  $M : \gamma \rightarrow \mathbb{SU}_2$ , these are equivalent to*

$$(1.12) \quad \rho_1(\lambda_0) = \pm 1, \quad \rho'_1(\lambda_0) = 0.$$

*Proof.* Since  $M(\lambda_0) \in \mathbb{U}_2$ ,  $M(\lambda_0)$  is a multiple of  $I$  iff  $\rho_1(\lambda_0) = \rho_2(\lambda_0)$ . Assuming this, differentiating the characteristic equation  $\rho^2 - (\text{tr } M)\rho + \det M = 0$  twice and evaluating at  $\lambda_0$  yields

$$(1.13) \quad \rho'^2(\lambda_0) - (\text{tr } M'(\lambda_0))\rho'(\lambda_0) + \det M'(\lambda_0) = 0.$$

Hence  $\rho'_1(\lambda_0), \rho'_2(\lambda_0)$  are the eigenvalues of  $M'(\lambda_0)$ . But if  $\text{tracefree}(M'(\lambda_0)) = 0$ , then the eigenvalues of  $M'(\lambda_0)$  are equal.

Conversely, since the eigenvalues of  $M'(\lambda_0)$  are  $\rho_j(\lambda_0)$ , we have by equation (1.13)

$$(1.14) \quad M'(\lambda_0)^2 - (\text{tr } M'(\lambda_0))M'(\lambda_0) + (\det M'(\lambda_0))M'(\lambda_0) = 0.$$

If  $\rho'_1(\lambda_0) = \rho'_2(\lambda_0)$ , then  $(\text{tr } M'(\lambda_0))^2 = 4 \det M'(\lambda_0)$  and equation (1.14) becomes  $(\text{tracefree}(M'(\lambda_0)))^2 = 0$ . Differentiating  $MM^* = I$  shows that  $\rho_1(\lambda_0)^{-1}M'(\lambda_0)$  is skew-hermitian. It follows that  $M'(\lambda_0)$  a multiple of  $I$ .  $\square$

### 1.3. Gauge equivalence.

**Notation 1.9.** Let  $\Sigma$  be a Riemann surface,  $\xi \in \Omega_\Sigma^1(\Lambda_r^{-1}\mathfrak{gl}_2(\mathbb{C}))$  and  $g : \tilde{\Sigma} \rightarrow \Lambda_r\text{GL}_2(\mathbb{C})$  and suppose that the monodromy group of  $g$  is a subgroup of  $\mathbb{C}^*I$ . The gauged potential  $\xi.g$  is

$$\xi.g = g^{-1}\xi g + g^{-1}dg.$$

If  $\Phi$  is a solution to the ODE  $d\Phi = \Phi\xi$ , then  $\Psi = \Phi g$  is a solution to the gauged ODE  $d\Psi = \Psi(\xi.g)$ .

The following lemma provides the basic facts relating to gauge equivalence.

**Lemma 1.10.** *Let  $\Sigma, \tilde{\Sigma}, \Gamma, r, \xi, \Phi$  and  $M_\Phi$  be as in notation 1.6. Let  $g : \tilde{\Sigma} \rightarrow \Lambda_r^+ \mathrm{GL}_2(\mathbb{C})$  (resp.  $\Lambda_r^+ \mathrm{GL}_2(\mathbb{C})$ ) and suppose that the monodromy of  $g$  takes values in  $\mathbb{C}^* \mathbf{I}$ . Let  $M_{\Phi g}$  be the monodromy of  $\Phi g$ . Then*

- (i)  $\xi.g \in \Omega_\Sigma^1(\Lambda_r^{-1} \mathrm{sl}_2(\mathbb{C}))$  (resp.  $\Omega_\Sigma^1(\Lambda_r^{-1} \mathrm{sl}_2(\mathbb{C}))$ ).
- (ii)  $M_{\Phi g} = cM_\Phi$ , where  $c$  is the monodromy of  $g$ .
- (iii)  $\mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi]] = \mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi g]]$ .

*Proof.* To show (i), an examination of the series for  $g$  and  $\xi$  in  $\lambda$  at  $\lambda = 0$  show that  $\xi.g$  is holomorphic at  $\lambda = 0$ , hence  $\xi.g$  is holomorphic in  $\mathcal{D}_r$ . In the case  $\Omega_\Sigma^1(\Lambda_r^{-1} \mathrm{sl}_2(\mathbb{C}))$ , note that if  $\det g = \mathbf{I}$  and  $\xi$  is tracefree, then  $\xi.g$  is tracefree.

Proof of (ii):

$$\begin{aligned} M_{\Phi g}(\tau) &= (\tau^*(\Phi g))(\Phi g)^{-1} = (\tau^*\Phi)((\tau^*g)g^{-1})\Phi^{-1} \\ &= c(\tau)(\tau^*\Phi)\Phi^{-1} = c(\tau)M_\Phi(\tau). \end{aligned}$$

*Proof of (iii).* Let  $\Phi = FB$  be the  $r$ -Iwasawa factorization of  $\Phi$ . Let  $(BG)(0) = UT$  be the pointwise Iwasawa factorization of  $(BG)(0)$  (so  $U \in \mathrm{U}_2$  and  $T \in \mathcal{T}^{\mathbb{R}} \mathrm{GL}_2(\mathbb{C})$ ). Then the  $r$ -Iwasawa factorization of  $\Phi g$  is

$$\Phi g = (FU)(U^{-1}Bg),$$

and

$$\begin{aligned} \mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi g]] &= \mathrm{Sym}_\lambda[FU] = -2H^{-1}(FU)'(FU)^{-1} \\ &= -2H^{-1}F'F^{-1} = \mathrm{Sym}_\lambda[F] = \mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi]]. \end{aligned}$$

□

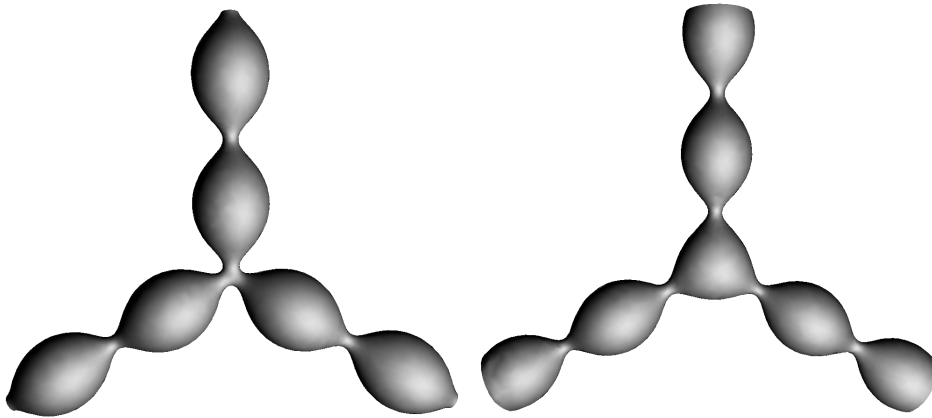


FIGURE 2. A pair of CMC trinoids with unduloid ends (neck-sizes  $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ ). They have respectively a central neck and central bulge, exhibiting a phase shift.

## 2. DELAUNAY IMMERSIONS

**2.1. Delaunay surfaces via DPW.** CMC surfaces whose ends are asymptotic to Delaunay surfaces can be constructed as local perturbations of a base Delaunay surface. Hence Delaunay surfaces are first discussed.

The only CMC surfaces of revolution are the round cylinder, the Delaunay unduloids (embedded Delaunay surfaces), the sphere, and the Delaunay nodoids (immersed non-embedded Delaunay surfaces).

**Definition 2.1.** Let  $f$  be an conformal CMC immersion of constant mean curvature  $H$  which is a surface of revolution (a sphere, cylinder or Delaunay surface). The *necksize*  $n$  of  $f$  is the minimum radius of the foliating circles, taken to be negative in the case of nodoids. The *weight* of  $f$  is  $w = 4n(H^{-1} - n)$ .

In the case  $H = 1$ , the round cylinder has weight 1 and necksize  $\frac{1}{2}$ , the unduloids have weight in  $(0, 1)$  and necksize in  $(0, \frac{1}{2})$ , the round sphere has weight and necksize 0, and the nodoids have weight and necksize in  $(-\infty, 0)$ .

The DPW construction of Delaunay surfaces [7] are as follows.

**Lemma 2.2.** Let  $A \in \Lambda_1^{-1}\mathrm{sl}_2(\mathbb{C})$  satisfy  $A^* = A$ , so that

$$(2.1) \quad A = \begin{pmatrix} c & a\lambda^{-1} + \bar{b} \\ b + \bar{a}\lambda & -c \end{pmatrix} \quad a, b \in \mathbb{C}^*, c \in \mathbb{R},$$

Let  $\Phi : \mathbb{C} \times \mathcal{A}_0 \rightarrow \mathrm{SL}_2(\mathbb{C})$  be defined by  $\Phi = \exp(\zeta A)$ . Then

(i)  $f_\lambda = \mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi]]$  is independent of the choice of  $r \in (0, 1]$ . For each  $\lambda \in \mathbb{S}^1$ ,  $f_\lambda$  has screw symmetry.

(ii) Let  $\mu(\lambda)$  be an eigenvalue of  $A$ . If

$$(2.2) \quad \mu(1) = \pm \frac{1}{2}, \quad \mu'(1) = 0,$$

then  $f_1$  satisfies  $f_1(\zeta + 2\pi i) = f_1(\zeta)$  and is a once-wrapped conformal immersion of a Delaunay surface with weight  $16abH^{-2}$ . The eigenvalues of the monodromy  $M = \exp(2\pi i A)$  of  $\Phi$  are  $\exp(\pm 2\pi i \mu_w)$ , where  $w = 16ab \in (-\infty, 1] \setminus \{0\}$  and

$$(2.3) \quad \mu_w = \frac{1}{2} \sqrt{1 + \frac{w(\lambda - 1)^2}{4\lambda}}.$$

(iv) If  $A_1, A_2$  are of the form (2.1) with  $\det A_1 = \det A_2$ , and  $f_1 = \mathrm{Sym}[\mathrm{Uni}_r[\exp(zA_1)]]$  and  $f_2 = \mathrm{Sym}[\mathrm{Uni}_r[\Phi(zA_2)]]$ , then there exists an isometry  $T : \mathrm{su}_2 \rightarrow \mathrm{su}_2$  and a coordinate change  $z = \tilde{z} + c$ ,  $c \in \mathbb{R}$  such that  $f_2(z) = T(f_1(\tilde{z}))$ .

*Proof.*  $f$  is independent of the choice of  $r$  by lemma 1.5. Let  $\theta \in \mathbb{R}$ ,  $u = \exp(i\theta)$  and  $U = \exp(i\theta A)$ . Then  $u \in \mathrm{SU}_1$ ,  $U \in \Lambda_r^* \mathrm{SL}_2(\mathbb{C})$ . Let  $F = \mathrm{Uni}_r[\Phi]$ . Then

$$\begin{aligned}\Phi(uz) &= U\Phi(z) \\ F(uz) &= UF(z) \\ f(uz) &= \mathrm{Ad}_U f(z) - \frac{2}{H}U'U^{-1}.\end{aligned}$$

A calculation shows that there exists  $S, T : \mathbb{S}^1 \rightarrow \mathfrak{su}_2$  such that

$$f_1(uz) + S = \mathrm{Ad}_{U(1)}(f_1(z) + S) + T.$$

This implies that  $f$  has screw symmetry, and is hence an associate family of Delaunay immersions. The monodromy of this solution,  $M_\Phi = \exp(2\pi i A)$ , satisfies the closing condition (1.5). Under the hypotheses on the eigenvalues,  $M_\Phi$  satisfies the closing condition (1.6) at  $\lambda_0 = 1$ , so  $f_1$  is monodromy-free along a loop around  $z = 0$ . A calculation shows that the weight of  $f_1$  is  $16ab$ . The proof of (iv) is omitted.  $\square$

## 2.2. Dressed Delaunay immersions.

**Lemma 2.3.** *Let  $\gamma$  be an open segment of  $\mathbb{S}^1$ ,  $\lambda_0 \in \gamma$ ,  $\gamma^* = \gamma \setminus \{\lambda_0\}$  and  $M : \gamma^* \rightarrow \mathrm{U}_2$  (resp.  $\mathrm{SU}_2$ ) a real analytic map which extends meromorphically to a neighborhood of  $\gamma$ . Then  $M$  extends to a real analytic map  $M : \gamma \rightarrow \mathrm{U}_2$  (resp.  $\mathrm{SU}_2$ ).*

*Proof.* Since  $M$  takes values in  $\mathrm{U}_2$  on  $\gamma^*$ , its entries are bounded in absolute value by 1 there. Since a meromorphic function at a pole is unbounded along every curve into the pole, the entries of  $M$  cannot have poles at  $\lambda_0$ . Hence  $M$  extends real analytically to  $\lambda_0$ .

Since  $MM^* = \mathrm{I}$  on  $\gamma^*$ , then  $MM^* = \mathrm{I}$  on  $\gamma$  by the continuity of  $MM^*$ . If  $\det M(\lambda_0) = 1$  on  $\gamma^*$ , then  $\det M = \mathrm{I}$  on  $\gamma$  by the continuity of  $\det M$ .  $\square$

Lemma 2.5 provides a “unitary-commutator” factorization theorem, used in lemma 2.6.

**Notation 2.4.** For  $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathrm{M}_{2 \times 2}(\mathbb{C})$ , define  $\widehat{X} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$ ,

so  $X\widehat{X} = (\det X)\mathrm{I}$ .

**Lemma 2.5.** *Let  $M \in \Lambda_1^* \mathrm{SL}_2(\mathbb{C})$ . Let  $C : \mathbb{S}^1 \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  be a real analytic map with  $\det C \not\equiv 0$  such that the extension of  $CMC^{-1}$  across  $\{\det C = 0\}$  is in  $\Lambda_1^* \mathrm{SL}_2(\mathbb{C})$  (see lemma 2.3). Then there exists  $U \in \Lambda_1^* \mathrm{SL}_2(\mathbb{C})$  and  $R : \mathbb{S}^1 \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  such that  $C = UB$  and  $[R, M] = 0$ .*

*Proof.* Since  $C \neq 0$ , there exists  $c \in \mathbb{C}^*$  such that  $V = cC + \widehat{cC}^* \neq 0$ . Then  $V = \widehat{V}^*$ . It follows that  $\det V$  takes values in  $\mathbb{R}^{\geq 0}$  and  $\det V \neq 0$ . Hence there exists a well-defined non-negative square root  $\sqrt{\det V}$  on  $\mathbb{S}^1$  which is not identically 0.

Define  $U = (\det V)^{-1/2}V$  away from the zeros of  $\det V$ . By lemma 2.3(i),  $U$  extends analytically to  $\mathbb{S}^1$  and  $U \in \Lambda_1^* \mathrm{SL}_2(\mathbb{C})$ .

Define  $R = U^{-1}C$ . Then  $CMC^{-1}C - CM = 0$  and, using the fact that  $M$  and  $CMC^{-1}$  satisfy the reality condition,  $CMC^{-1}\widehat{C}^* - \widehat{C}^*M = 0$ . Hence  $CMC^{-1}U - UM = 0$ , and so

$$[R, M] = [U^{-1}C, M] = U^{-1}(CMC^{-1}U - UM)U^{-1}C = 0.$$

Hence  $U, R$  satisfy the conditions of the lemma.  $\square$

Lemma 2.6 shows that under suitable conditions, a dressed Delaunay immersion is ambient isometric to the original Delaunay immersion.

**Lemma 2.6.** *Let  $A \in \Lambda_1^{-1} \mathrm{sl}_2(\mathbb{C})$  satisfy  $A = A^*$ ,  $\Phi = \exp(\zeta A)$  and  $f_\lambda = \mathrm{Sym}_\lambda[\mathrm{Uni}_r[\Phi]]$  the Delaunay associate family. Let  $C \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ , and suppose that  $C$  is the boundary of an analytic map  $C : \{r < |\lambda| < 1 + \epsilon\} \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  for some  $\epsilon \in \mathbb{R}^+$  such that  $\{\det C = 0\} \subset \mathbb{S}^1$ . Suppose that  $C \exp(2\pi i A) C^{-1}$  satisfies the reality condition on  $\mathbb{S}^1 \setminus \{\det C = 0\}$ . Let  $\tilde{f}_\lambda = \mathrm{Sym}_\lambda[\mathrm{Uni}_r[C\Phi]]$ . Then (i) There exists  $\tilde{A}$  of the form (2.1),  $U_0 \in \Lambda_r^* \mathrm{SL}_2(\mathbb{C})$  and  $B_0 \in \Lambda_r^+ \mathrm{GL}_2(\mathbb{C})$  such that  $C\Phi = U_0 \exp(\zeta \tilde{A}) C_+$ . (ii) Then there exists  $c \in \mathbb{R}^+$  and an isometry  $T$  of  $\mathrm{su}_2$  such that  $\tilde{f}_\lambda(\zeta) = T(f_\lambda(\zeta + c))$ .*

*Proof.* Let  $C_u C_+$  be the  $r$ -Iwasawa factorization of  $C$ . Because  $C_u$  is analytic on  $\mathcal{A}_r$  with  $\det C_u = 1$ ,  $C_+$  is the boundary of an analytic map  $C_+ : \{r < |\lambda| < 1 + \epsilon\} \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  such that  $\{\det C_+ = 0\} \subset \mathbb{S}^1$ . By lemma 2.5, there exist analytic maps  $U \in \Lambda_1^* \mathrm{SL}_2(\mathbb{C})$  and  $R : \mathbb{S}^1 \rightarrow \mathrm{M}_{2 \times 2}(\mathbb{C})$  such that  $C_+ = UR$  and  $[R, A] = 0$ .  $U$  and  $R$  can be extended to  $\mathcal{A}_s$  for some  $s \in (r, 1)$ .

Then  $\tilde{A} = UAU^{-1} = C_+AC_+^{-1}$  on  $\{s < |\lambda| < 1\}$ . But  $UAU^{-1}$  extends analytically to  $\mathcal{A}_s$ , and  $C_+AC_+^{-1}$  extends holomorphically to  $\{0 < |\lambda| < 1\}$  and meromorphically to 0 with a simple pole in the upper right entry. Moreover, this extension satisfies  $\tilde{A} = \tilde{A}^*$ , since  $UAU^{-1} = (UAU^{-1})^*$ . It follows that  $\tilde{A}$  is of the form (2.1).

On  $\mathcal{C}_r$ ,

$$C\Phi = C_u \exp(\zeta \tilde{A}) C_+.$$

Hence  $\mathrm{Sym}_\lambda[\mathrm{Uni}_r[C\Phi]]$  and  $\mathrm{Sym}_\lambda[\mathrm{Uni}_r[\exp(\zeta \tilde{A})]]$  are the same surface up to rigid motion. The result follows by lemma (2.2)(iv).  $\square$

**2.3. Delaunay asymptotics.** The following lemma estimates the growth rate of the gauge  $B$  which gauges the Maurer-Cartan form for the Delaunay associate family to the Delaunay DPW potential. Since the estimate is for  $|\lambda|$  near 1, the explicit Delaunay frame is not required; the growth rate can be estimated by using the periodicity in the axial direction. The result is essentially that  $B(x + iy)$  grows in the axial direction  $x$  like  $e^{c|x|}$ , where  $c$  is the maximum value of the Delaunay eigenvalue on  $\mathbb{S}^1$ . The estimate is used in the asymptotics theorem 3.4 showing that a perturbation of the DPW Delaunay potential is asymptotically Delaunay.

In the following,  $|X|$  denotes the matrix 2-norm, and for  $r \in (0, 1]$ ,

$$\|X(\lambda)\|_r = \max_{\lambda \in \mathbb{C}_r} |X(\lambda)|.$$

**Lemma 2.7.** *Let  $\Sigma = \mathbb{C}$ . Let  $A$  be a Delaunay residue (equation (2.1)), let  $\Phi = \exp(\zeta A)$ , let  $C \in \Lambda_{\uparrow 1}^{+, \mathbb{R}} M_{2 \times 2}(\mathbb{C})$  and let  $C\Phi = FB$  be the  $r_0$ -Iwasawa factorization of  $C\Phi$  for some  $r_0 \in (0, 1]$ , and extend  $B$  to  $\Sigma \times \mathcal{D}_1$  as in lemma 1.5. Let  $\mu$  be an eigenvalue of  $A$  and let  $c = \|\operatorname{Re} \mu\|_1$ . Then there exists  $c_0 \in \mathbb{R}^+$  such that for all  $\epsilon > 0$ , there exists  $R(\epsilon) \in (0, 1)$  such that*

$$\|B(\zeta, \lambda)\|_r \leq c_0 \exp((c + \epsilon)|\operatorname{Re} \zeta|)$$

for all  $\zeta \in \mathbb{C}$  and all  $r \in (R(\epsilon), 1]$ .

*Proof.* First we prove the theorem in the case  $C = I$ . With  $\zeta = x + iy$ , the screw symmetry of the Delaunay family implies that  $F$  decouples into  $x$ - and  $y$ -dependent factors  $F = \exp(iyA)F_1(x)$  for some  $F_1 : \Sigma \rightarrow \Lambda_1^* \operatorname{SL}_2(\mathbb{C})$ . Then  $B(x) = F_1(x)^{-1} \exp(xA)$  can be estimated by estimating  $\exp(xA)$  and  $F_1(x)$ .

*Step 1: estimate  $\exp(xA)$ .* From the formula

$$\exp(xA) = \frac{1}{2}e^{x\mu}(I + \mu^{-1}A) + \frac{1}{2}e^{-x\mu}(I - \mu^{-1}A)$$

we obtain the pointwise estimate

$$|\exp(xA)| \leq (\max |I \pm \mu^{-1}A|) \exp(|\operatorname{Re} \mu||x|),$$

for all  $x \in \mathbb{R}$  and all  $\lambda$  at which  $\max |I \pm \mu^{-1}A|$  is finite. Since  $\max \|I \pm \mu^{-1}A\|_r$  is continuous and finite at  $r = 1$ , there exists  $R_1 \in (0, 1)$  such that  $\max \|I \pm \mu^{-1}A\|_r$  is finite for all  $r \in [R_1, 1]$ . Then

$$c_1 = \sup_{r \in [R_1, 1]} \max(\|I \pm \mu^{-1}A\|_r).$$

is finite. Then for all  $x \in \mathbb{R}$  and all  $r \in (R_1, 1]$ ,

$$\|\exp(xA)\|_r \leq c_1 \exp(\|\operatorname{Re} \mu\|_r |x|).$$

The continuity of  $\|\operatorname{Re} \mu\|_r$  at  $r = 1$  together with  $\|\operatorname{Re} \mu\|_1 = c$  imply that all  $\epsilon > 0$  there exists  $R \in (0, 1)$  such that for all  $r \in (R, 1]$ ,  $\|\operatorname{Re} \mu\|_r < c + \epsilon$ . Hence for all  $\epsilon > 0$  there exists  $R \in (0, 1)$  such that for all  $x \in \mathbb{R}$  and all  $r \in (R, 1]$ ,

$$(2.4) \quad \|\exp(xA)\|_r \leq c_1 \exp((c + \epsilon)|x|).$$

*Step 2: estimate  $F_1(x)$ .*  $F_1$  is periodic in axial direction the sense that there exist  $\rho \in \mathbb{R}$  and  $M \in \Lambda_1^* \operatorname{SL}_2(\mathbb{C})$  such that

$$F_1(x_0 + n\rho) = M^n F(x_0)$$

for all  $x \in \mathbb{R}$  and all  $n \in \mathbb{Z}$ . There exists  $R_2 \in (0, 1)$  such that  $\|F(x, \lambda)\|_r$  is finite for all  $r \in [R_2, 1]$  and all  $x \in \mathbb{R}$ . Then

$$c_2 = \sup_{(x, r) \in [0, \rho) \times [R_2, 1]} \|F(x, \lambda)\|_r$$

is finite.

Given any  $x \in \mathbb{R}$ , there exists  $x_0 \in [0, \rho)$  and  $n \in \mathbb{Z}$  such that  $x = x_0 + n\rho$ . Hence

$$\|F_1(x)\|_r \leq c_2 (\|M\|_r)^n.$$

The continuity of  $\|M\|_r$  at  $r = 1$  together with  $\|M\|_1 = 1$  imply that for all  $\epsilon' > 0$  there exists  $R$  such that for all  $r \in (R, 1)$ ,  $\|M\|_r < 1 + \epsilon'$ . Given  $\epsilon > 0$ , let  $\tilde{\epsilon} = \min(\epsilon, 1)$ , and choose  $\epsilon' = \exp(\rho\tilde{\epsilon})$ . Then there exists  $R$  such that for all  $r \in (R, 1)$ ,  $\|M\|_r < 1 + \epsilon' = \exp(\rho\tilde{\epsilon})$ . Hence

$$(\|M\|_r)^{|n|} < \exp(\rho\tilde{\epsilon}|n|) = \exp(\tilde{\epsilon}|x - x_0|) \leq \exp(\tilde{\epsilon}\rho) \exp(\tilde{\epsilon}|x|).$$

Hence with  $c_3 = c_2 \exp(\rho)$ ,

$$(2.5) \quad \|F_1(x)\|_r \leq c_3 \exp(\epsilon|x|).$$

*Step 3: estimate  $B$ .*  $B(x) = F_1^{-1}(x) \exp(xA)$ , so

$$\|B(x, \lambda)\|_r \leq \|F_1(x, \lambda)\|_r \|\exp(xA(\lambda))\|_r.$$

Given  $\epsilon > 0$ , by (2.4) and (2.5) we can choose  $R$  such that for all  $x \in \mathbb{R}$  and all  $r \in (R, 1]$ ,

$$\|\exp(xA)\|_r \leq c_1 \exp((c + \epsilon/2)|x|)$$

and

$$\|F_1(x)\|_r \leq c_3 \exp((\epsilon/2)|x|).$$

Then with  $c_4 = c_1 c_3$ ,

$$\|B(x, \lambda)\|_r \leq c_4 \exp((c + \epsilon)|x|).$$

Now we prove the theorem for general  $C$ . By theorem 2.6,  $C\Phi = U_0 \tilde{\Phi} B_0$ , where  $\tilde{\Phi} = \exp(\zeta \tilde{A})$ ,  $U_0 \in \Lambda_r^* \operatorname{SL}_2(\mathbb{C})$ ,  $B_0 \in \Lambda_r \operatorname{GL}_2(\mathbb{C})$ . Let

$\tilde{\Phi} = \tilde{F}\tilde{B}$  be the Iwasawa factorization of  $\tilde{\Phi}$ . Then  $C\Phi = (U_0\tilde{F})(\tilde{B}B_0)$  is the Iwasawa factorization of  $C\Phi$ .

Then since  $B_0$  is  $\zeta$ -independent, for any  $\epsilon > 0$ , there exists  $R \in (0, 1)$  such that for all  $x \in \mathbb{R}$  and all  $r \in (R, 1]$ ,

$$\|\tilde{B}(x, \lambda)B_0(\lambda)\|_r \leq \|\tilde{B}(x, \lambda)\|_r \|B_0(\lambda)\|_r \leq c_0 \exp((c + \epsilon)|x|),$$

where  $c_0 = c_4 \sup_r \|B_0\|_r$ .  $\square$

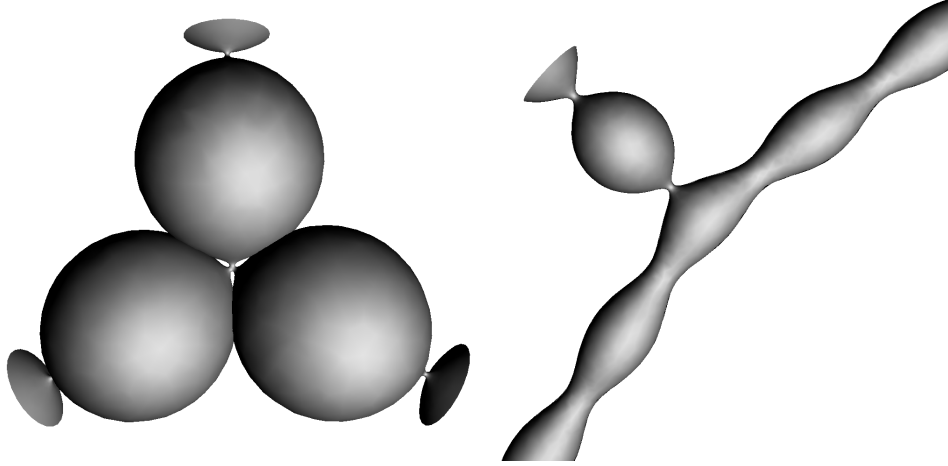


FIGURE 3. Trinoids with small necks. The lobes of the left example (necksizes  $(\frac{1}{40}, \frac{1}{40}, \frac{1}{40})$ ) intersect, making the surface Alexandrov embedded. The example on the right (necksizes  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{12})$ ) can be viewed as a Delaunay surface with a small-necked Delaunay surface with a small-necked Delaunay end added. The Delaunay surface bends slightly to balance the added end.

### 3. PERTURBATIONS OF DELAUNAY IMMERSIONS

**3.1. Perturbations at a simple pole.** The following lemma extends a basic result in ODE theory to the context of loops.

**Lemma 3.1.** *Let  $r \in (0, 1)$ . Let  $\xi_0, \xi \in \Omega_{\Sigma^*}^1(\Lambda_r^{-1}\mathrm{sl}_2(\mathbb{C}))$  be potentials with expansions in  $z$  at  $z = 0$*

$$\xi_0 = A \frac{dz}{z}, \quad \xi = A \frac{dz}{z} + Bdz + \mathcal{O}(z^1)dz.$$

*Let  $\mu$  be an eigenvalue of  $A$  and suppose that either*

- (i)  $\mu \notin \frac{1}{2}\mathbb{Z}^*$  along  $\mathbb{C}_r$ , or
- (ii)  $\mu \notin \frac{1}{2}\mathbb{Z}^* \setminus \{\pm\frac{1}{2}\}$  along  $\mathbb{C}_r$  and  $[A, B] = 0$ .

Then in a neighborhood  $U$  of  $p$  there exists a unique analytic map  $P : U \times \rightarrow \Lambda_r \mathrm{SL}_2(\mathbb{C})$  such that

$$(3.1) \quad \xi = \xi_0 \cdot P, \quad P(0, \lambda) = \mathrm{I}.$$

In the case  $[A, B] \equiv 0$ ,  $P = \mathrm{I} + Bz + \mathrm{O}(z^2)$ .

*Proof.* In case (i), a unique solution to (3.1) exists by a standard result in ODE theory,

In the case (ii), if  $\mu(\lambda_0) = \pm \frac{1}{2}$ , a calculation of the series  $P = \sum_{k=0}^{\infty} P_k z^k$  shows that the  $P_k$  are holomorphic in  $\mathcal{C}_r$ , and  $P_1 = B$ .  $\square$

**3.2. Gauging away the constant term.** Lemma 3.2 constructs a gauge and coordinate change which removes the constant term in a perturbed Delaunay potential.

**Lemma 3.2.** *Let  $r \in (0, 1]$ . Let  $\Sigma$  be a Riemann surface and  $p \in \Sigma$ . Let  $\xi \in \Omega_{\Sigma}^1(\Lambda_r^{-1} \mathrm{sl}_2(\mathbb{C}))$  with expansion*

$$(3.2) \quad \xi \cdot g = \xi_{-1} \frac{dz}{z} + \xi_0 dz + \mathrm{O}(z^1) dz.$$

Let  $\mu$  be an eigenvalue of  $\xi_{-1}$  and suppose

- (i)  $\mathrm{res}_{\lambda=0} \mu^2 \neq 0$
- (ii) for every  $\lambda_0 \in \mathcal{D}_r$ , if  $\mu(\lambda_0) \in \{\pm \frac{1}{2}\}$ , then  $\xi_0|_{\lambda=\lambda_0} = 0$ .

Then there exists a neighborhood  $U \in \Sigma$  of  $p$ , an analytic map  $g : U \times \mathcal{D}_1 \rightarrow \mathrm{GL}_2(\mathbb{C})$  such that  $g(z, 0)$  takes values in  $\mathcal{TGL}_2(\mathbb{C})$ , and a conformal coordinate  $\tilde{z} : U \rightarrow \mathbb{C}$  with  $\tilde{z}(p) = 0$  such that the expansion of  $\xi \cdot g$  in  $\tilde{z}$  at  $\tilde{z} = 0$  is

$$(3.3) \quad \xi \cdot g = \xi_{-1} \frac{d\tilde{z}}{\tilde{z}} + \mathrm{O}(\tilde{z}^1) d\tilde{z}.$$

*Proof.* For any  $k \in \mathbb{C}$ , define  $g_1$  and  $g$  by

$$\begin{aligned} u &= 4\mu^2 - 1, & v \mathrm{I} &= \xi_{-1} \xi_0 + \xi_0 \xi_{-1}, \\ g_1 &= (k - 2u^{-1}v)\xi_{-1} + u^{-1}(\xi_0 - [\xi_{-1}, \xi_0]), & g &= \mathrm{I} + g_1 z. \end{aligned}$$

A calculation shows that

$$(\mathrm{I} + \mathrm{ad}_{\xi_{-1}})g_1 = k\xi_{-1} - \xi_0,$$

from which it follows that

$$\xi \cdot g = \xi_{-1} \frac{dz}{z} + k\xi_{-1} dz + \mathrm{O}(z^1) dz.$$

Assumption (i) implies that  $u^{-1}v$  is holomorphic at  $\lambda = 0$  so  $k = \lim_{\lambda \rightarrow 0} 2u^{-1}v$  exists and is finite. With this choice of  $k$ , a calculation shows  $g_1$  is holomorphic at  $\lambda = 0$ . Assumption (ii) implies that  $g_1$  is

holomorphic in  $\mathcal{D}_r^*$ , and hence in  $\mathcal{D}_r$ . A calculation shows that  $g_1(0) \in TGL_2(\mathbb{C})$  and hence  $g(z, 0)$  takes values in  $TGL_2(\mathbb{C})$ .

Since  $g(0, \lambda) = I$ , a continuity argument shows that  $\det g \neq 0$  in a sufficiently small neighborhood of  $z = 0$ . In the coordinate  $\tilde{z}$  defined by  $z = \tilde{z} - k\tilde{z}^2$  in a neighborhood of  $z = 0$ ,  $\xi$  has the expansion (3.3)  $\square$

**3.3. Monodromy at simple poles.** The following lemma computes the eigenvalues of the monodromy of a perturbed potential  $\xi$  at a simple pole in terms of the residue of  $\xi$ .

**Lemma 3.3.** *Let  $r \in (0, 1)$ . Let  $\xi \in \Omega_{\Sigma^*}^1(\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}))$  be a potential with expansions in  $z$  at  $z = 0$   $\xi = Adz/z + O(z^0)dz$ , and let  $\mu$  be an eigenvalue of  $A$ . Suppose  $\xi$  satisfies condition (i) or (ii) of lemma 3.1. Then*

(i) *If  $\Phi : \tilde{\Sigma} \times \mathcal{C}_r \rightarrow GL_2(\mathbb{C})$  is a solution to the ODE  $d\Phi = \Phi\xi$ , and  $M$  is the monodromy of  $\Phi$  at  $z = 0$ , then the eigenvalues of  $M$  are  $\exp(\pm 2\pi i\mu)$ .*

(ii) *If  $\Phi : \tilde{\Sigma} \times \mathcal{C}_r \rightarrow M_{2 \times 2}(\mathbb{C})$  is a solution to the ODE  $d\Phi = \Phi\xi$  with  $\det \Phi \neq 0$ , and the monodromy  $M$  of  $\Phi$  at  $z = 0$  extends analytically to  $\mathcal{C}_r$  across  $\{\det \Phi = 0\}$ , then the eigenvalues of  $M$  are  $\exp(\pm 2\pi i\mu)$ .*

*Proof.* Since (ii) implies (i) we prove (ii). Let  $\xi_0 = Adz/z$ . By lemma 3.1 there exists a unique analytic map  $P : U \rightarrow \Lambda_r SL_2(\mathbb{C})$  such that  $\xi = \xi_0.P$  and  $P(0, \lambda) = I$ . Then there exists an analytic map  $C : \mathcal{C}_r \rightarrow M_{2 \times 2}(\mathbb{C})$  such that  $\Phi = C \exp((\log(z)A)P)$ .

Then  $M = C \exp(2\pi iA)C^{-1}$  on  $\mathcal{C}_r \setminus \{\det \Phi = 0\}$ , so the eigenvalues of  $M$  are  $\exp(2\pi i\mu)$  on  $\mathcal{C}_r \setminus \{\det \Phi = 0\}$ . Since by hypothesis  $M$  extends analytically to  $\mathcal{C}_r$ , the eigenvalues of  $M$  extend analytically to  $\mathcal{A}_r$ , and hence are  $\exp(2\pi i\mu)$  on  $\mathcal{C}_r$ .  $\square$

**3.4. Perturbed Delaunay asymptotics.** In this section it is shown that the immersion obtained from a suitable perturbation of a Delaunay potential is asymptotic to the base half-Delaunay surface (theorem 3.4).

In the following,  $|X|$  denotes the matrix 2-norm, and for a compact set  $S \subset \mathbb{C}^*$ ,

$$\|X(\lambda)\|_S = \max_{\lambda \in S} |X(\lambda)|.$$

The asymptotics theorem below shows that under certain conditions, the CMC immersion produced by a perturbation of a Delaunay potential is asymptotic to a half Delaunay surface.

**Theorem 3.4** (Asymptotics theorem). *Let  $\Sigma$  be a punctured annular neighborhood of 0 and  $\tilde{\Sigma}^* \rightarrow \Sigma^*$  its universal cover. Let  $\xi_0 = Adz/z \in \Omega_{\Sigma^*}^1(\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}))$  where  $A \in \Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})$  is of the form (2.1). Let  $\mu$  be an eigenvalue of  $A$  and suppose  $\|\operatorname{Re} \mu\|_{\mathbb{S}^1} < k$  for some integer  $k \geq 1$ .*

Let  $\xi \in \Omega_{\widetilde{\Sigma}^*}^1(\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}))$  be a perturbation of  $\xi_0$  whose expansion of  $\xi$  at  $z = 0$  is

$$\xi = Az^{-1}dz + O(z^{2k})dz.$$

Let  $\Phi : \widetilde{\Sigma}^* \times \mathcal{A}_r \rightarrow \mathrm{GL}_2(\mathbb{C})$  satisfy  $d\Phi = \Phi\xi$ . Let  $\Phi_0 = \Phi P^{-1}$ , where  $P$  is the gauge of lemma 3.1 with  $\xi_0.P = \xi$  and  $P(0, \lambda) = \mathrm{I}$ . Let  $f_0 = \mathrm{Sym}[\mathrm{Uni}_s[\Phi_0]]$ . Then

$$(3.4) \quad \lim_{z \rightarrow 0} \|f - f_0\|_{\mathbb{S}^1} = 0$$

$$(3.5) \quad \lim_{z \rightarrow 0} \|df - df_0\|_{\mathbb{S}^1} = 0.$$

*Proof.* Let  $\Phi_0 = F_0 B_0$  and  $\Phi = FB$  be the  $r$ -Iwasawa factorizations of  $\Phi_0$  and  $\Phi$  respectively. By hypothesis, the monodromy of  $\Phi_0$  is  $r$ -unitary. It follows that the monodromy of  $\Phi$  is  $r$ -unitary, and that  $B_0$ ,  $B$  and  $F_0^{-1}F$  are monodromy-free on  $\Sigma$ .

By lemma 3.1 the expansion of  $P(z)$  at  $z = 0$  is

$$P(z) = \mathrm{I} + \sum_{j=2k}^{\infty} P_j z^j.$$

Then

$$B_0 P B_0^{-1} - \mathrm{I} = \sum_{j=2k}^{\infty} B_0 P_j B_0^{-1} z^j,$$

so

$$\|B_0 P B_0^{-1} - \mathrm{I}\|_{c_r} \leq \sum_{j=2k}^{\infty} \|B_0\|_{c_r} \|P_j\|_{c_r} \|B_0^{-1}\|_{c_r} |z|^j.$$

By hypothesis  $c = \|\mathrm{Re} \mu\|_{\mathbb{S}^1} < k$ . Let  $\epsilon \in (0, k-c)$ , so that  $l = c + \epsilon < k$ . By lemma 2.7, there exists  $R$  such that for all  $r \in (R, 1]$  and all  $z \in \Sigma$ ,  $|B_0| < c_0 |z|^{-l}$  for some constant  $c_0 \in \mathbb{R}^+$ . Then

$$\|B_0 P B_0^{-1} - \mathrm{I}\|_{c_r} \leq \sum_{j=2k}^{\infty} \|P_j\|_{c_r} |z|^{j-2l}.$$

By the choice of  $l$ , the exponent  $j - 2l > 0$  for all  $j \geq 2k$ , so

$$\lim_{z \rightarrow 0} \|B_0 P B_0^{-1} - \mathrm{I}\|_{c_r} = 0.$$

The holomorphicity of  $B_0 P B_0^{-1}$  in  $\lambda$  with Cauchy's integral formula implies

$$\lim_{z \rightarrow 0} \|(B_0 P B_0^{-1})'\|_{c_r} = 0.$$

With  $G = F_0^{-1}F = \text{Uni}_r[B_0PB_0^{-1}]$ ,

$$(3.6) \quad \lim_{z \rightarrow 0} \|G - I\|_A = 0$$

$$(3.7) \quad \lim_{z \rightarrow 0} \|G'\|_A = 0$$

$$(3.8) \quad \lim_{z \rightarrow 0} \|BB_0^{-1}\|_D = 0$$

for every compact subset  $A \subset \mathcal{A}_r$  and  $D \subset \mathcal{D}_r$ .

From the Sym formula (1.4) we get

$$f - f_0 = -2|H|^{-1}F_0G'F^{-1}.$$

Then since  $\|F_0\|_{\mathbb{S}^1} = 1$  and  $\|F^{-1}\|_{\mathbb{S}^1} = 1$  we have

$$\|f - f_0\|_{\mathbb{S}^1} \leq 2|H|^{-1}\|G'\|_{\mathbb{S}^1},$$

and equation (3.4) follows.

Differentiating the Sym formula (1.4) yields

$$df = -2H^{-1}F\Theta'F^{-1}, \quad df_0 = -2H^{-1}F_0\Theta'_0F_0^{-1},$$

where

$$\Theta' = \frac{1}{4}HvE, \quad \Theta'_0 = \frac{1}{4}Hv_0E_0,$$

$v^2dz \otimes d\bar{z}$  and  $v_0^2dz \otimes d\bar{z}$  are the metrics of  $f$  and  $f_0$  respectively,

$$E = -i|\alpha|^{-1} \begin{pmatrix} 0 & \alpha\lambda^{-1} \\ \bar{\alpha}\lambda & 0 \end{pmatrix}, \quad E_0 = -i|\alpha_0|^{-1} \begin{pmatrix} 0 & \alpha_0\lambda^{-1} \\ \bar{\alpha}_0\lambda & 0 \end{pmatrix},$$

and  $\alpha, \alpha_0$  are defined by

$$\xi = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \lambda^{-1} + O(\lambda^0)dz, \quad \xi_0 = \begin{pmatrix} 0 & \alpha_0 \\ 0 & 0 \end{pmatrix} \lambda^{-1} + O(\lambda^0)dz.$$

Then

$$df - df_0 = -\frac{1}{2}v_0F_0(v_0^{-1}vGE - E_0G)F^{-1}.$$

Then since  $\|F_0\|_{\mathbb{S}^1} = 1$  and  $\|F^{-1}\|_{\mathbb{S}^1} = 1$  we have

$$\begin{aligned} & \|df - df_0\|_{\mathbb{S}^1} \\ & \leq \frac{1}{2}\|v_0\|_{\mathbb{S}^1}\|v_0^{-1}vGE - E_0G\|_{\mathbb{S}^1} \\ & \leq \frac{1}{2}\|v_0\|_{\mathbb{S}^1} (\|v_0v^{-1}G - I\|_{\mathbb{S}^1}\|E\|_{\mathbb{S}^1} + \|G - I\|_{\mathbb{S}^1}\|E_0\|_{\mathbb{S}^1} + \|E - E_0\|_{\mathbb{S}^1}). \end{aligned}$$

$\|v_0\|_{\mathbb{S}^1}$ ,  $\|E\|_{\mathbb{S}^1}$  and  $\|E_0\|_{\mathbb{S}^1}$  are bounded on  $\Sigma$ , and

$$\lim_{z \rightarrow 0} \|E - E_0\|_{\mathbb{S}^1} = 0.$$

By equation (3.8) and remark 1.1,

$$\lim_{z \rightarrow 0} v_0^{-1}v = 1,$$

from which it follows, using equation (3.6), that

$$\lim_{z \rightarrow 0} \|v_0 v^{-1} G - I\|_{\mathbb{S}^1} = 0.$$

Equation (3.5) follows. □

**Corollary 3.5.** *If in theorem 3.4  $A$  satisfies equations (2.1)–(2.2), the weight  $w$  associated to  $A$  satisfies  $w > -3$ , the expansion of  $\xi$  is*

$$\xi = Az^{-1} dz + O(z^1) dz,$$

*and the monodromy  $\Phi$  at  $z = 0$  is in  $\Lambda_r^* \text{SL}_2(\mathbb{C})$ , by lemma 2.6  $f_0$  is a Delaunay associate family with weight  $w$ , the Delaunay and perturbed surfaces are closed at  $\lambda = 1$ , and the theorem shows  $C^1$  convergence of the perturbed surface to the Delaunay surface.*

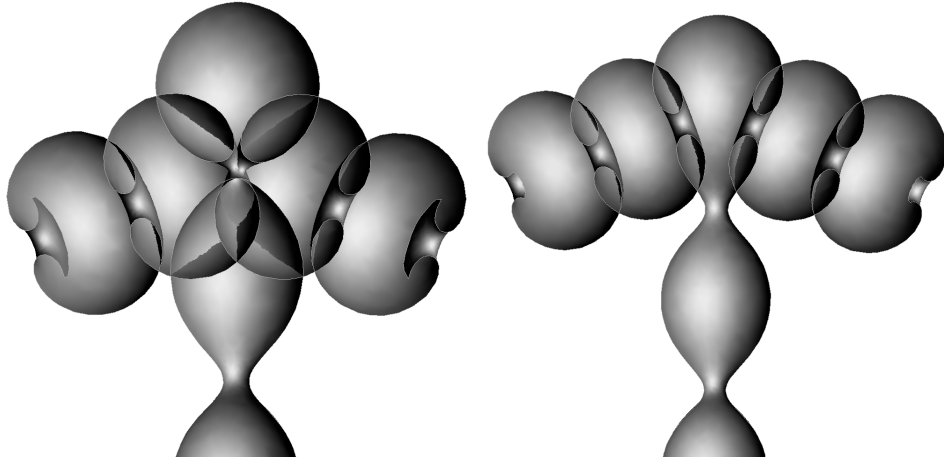


FIGURE 4. A pair of CMC trinoids with two nodoid ends (necksizes  $(\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$ ).

#### 4. UNITARIZATION OF LOOP GROUP MONODROMY REPRESENTATIONS

This section proves the “gluing theorem” (theorem 4.9): if a monodromy representation of the ODE (1.3) on the  $n$ -punctured Riemann sphere is unitarizable pointwise on  $\mathbb{S}^1$ , then the monodromy representation is unitarizable by a dressing matrix on an  $r$ -circle which is analytic in  $\lambda$ . The proof is based on lemmas 4.2–4.8.

**Notation 4.1.**  $M \in \text{GL}_2(\mathbb{C})$  is *unitarizable* if there exists  $C \in \text{GL}_2(\mathbb{C})$  such that  $CMC^{-1} \in \text{U}_2$ .

The set  $\mathcal{M} = \{M_1, \dots, M_n\} \subset \text{GL}_2(\mathbb{C})$  is *simultaneously unitarizable* iff for all  $j \in \{1, \dots, n\}$  there exists  $C \in \text{GL}_2(\mathbb{C})$  such that  $CM_j C^{-1} \in \text{U}_2$ .

$\mathcal{M}$  is *nondegenerate* iff  $[M_i, M_j] \neq 0$  for some pair  $i \neq j$ .

**4.1. Birkhoff factorizations.** Two Birkhoff factorizations are given for singular loops on  $\mathbb{S}^1$ : a scalar version (lemma 4.2) and a matrix version (lemma 4.3).

**Lemma 4.2.** *Let  $f : \mathbb{S}^1 \rightarrow \mathbb{R}^{\geq 0}$  be an analytic map with  $f \not\equiv 0$ . Then there exists an analytic map  $h : \mathbb{S}^1 \rightarrow \mathbb{C}$  which is the boundary of an analytic map  $\mathcal{D}_1 \rightarrow \mathbb{C}^*$ , such that  $f = h^*h$ .*

*Proof.* Since  $f$  is real and non-negative, each of its zeros is of even order. Let  $\{a_1, \dots, a_n\} \subset \mathbb{S}^1$  be the zeros of  $f$ , each with multiplicity two, and let  $q = \prod_{j=1}^n (\lambda - a_j)$ . Then the function  $g = f/(q^*q)$  has no zeros on  $\mathbb{S}^1$  and satisfies  $g = g^*$ . Let

$$g = r\lambda^p g_- g_+$$

be the (rank 1) Birkhoff factorization of  $g$ , such that  $g_+$  extends analytically without zeros to  $\overline{\mathcal{D}}_1$ ,  $g_-$  extends analytically without zeros to  $\overline{\mathcal{D}}_1'$ , and normalized with  $r \in \mathbb{C}$ ,  $g_+(0) = 1$  and  $g_-(\infty) = 1$ . But  $g^* = g$  on  $\mathbb{S}^1$ , so on  $\mathbb{S}^1$  we have the equality

$$r\lambda^p g_- g_+ = \bar{r}\lambda^{-p} g_+^* g_-^*.$$

By the uniqueness of the Birkhoff factorization,  $g_- = g_+^*$ ,  $p = 0$  and  $r = \bar{r}$ . Since  $f$  is nonnegative on  $\mathbb{S}^1$ ,  $r$  is positive. Then the function

$$h = \sqrt{r}g_+ q$$

is analytic on  $\mathbb{S}^1$ , is the boundary of the map  $h : \mathcal{D}_1 \rightarrow \mathbb{C}^*$  and satisfies  $f = h^*h$ .  $\square$

**Lemma 4.3.** *Let  $X : \mathbb{S}^1 \rightarrow M_{2 \times 2}(\mathbb{C})$  be a positive semidefinite analytic map with  $\det X \not\equiv 0$ . Then there exists  $C \in \Lambda_{\mathbb{1}}^{+, \mathbb{R}} M_{2 \times 2}(\mathbb{C})$  and an analytic map  $f : \mathbb{S}^1 \rightarrow \mathbb{R}^{\geq 0}$  such that  $fX = C^*C|_{\mathbb{S}^1}$ .*

*Proof.* The map  $X$  can be written

$$X = \begin{pmatrix} x_1 & y \\ y^* & x_2 \end{pmatrix}$$

where the functions  $x_1, x_2$  satisfy  $x_1 = x_1^*$  and  $x_2 = x_2^*$ , are real-valued and non-negative on  $\mathbb{S}^1$ , and  $x_1 \not\equiv 0, x_2 \not\equiv 0$  on  $\mathcal{A}_r$ .

The function  $d = \det X$  satisfies  $\det X \not\equiv 0$  on  $\mathcal{A}_r$ , and since  $X$  is positive semidefinite,  $d$  is real-valued and non-negative on  $\mathbb{S}^1$ .  $d = e^*e$  be the singular Birkhoff factorizations of  $d$  (lemma 4.2). Let

$$Y = \begin{pmatrix} x_1 & y \\ 0 & e \end{pmatrix}.$$

Then  $Y$  is an analytic map on  $\mathbb{S}^1$  which satisfies

$$x_1 X = Y^* Y.$$

For some  $r \in (0, 1)$ ,  $X$  extends analytically to a map  $\tilde{X} : \mathcal{A}_r \rightarrow \mathbb{M}_{2 \times 2}(\mathbb{C})$  such that  $\tilde{X}_{11}$  and  $\det \tilde{X}$  have no zeros in  $\mathcal{A}_s \setminus \mathbb{S}^1$ . Then  $Y$  likewise extends analytically to a map  $\tilde{Y} : \mathcal{A}_r \rightarrow \mathbb{M}_{2 \times 2}(\mathbb{C})$  such that  $\det \tilde{Y}$  has no zeros in  $\mathcal{A}_r \setminus \mathbb{S}^1$ . Let  $\tilde{Y}|_{\mathcal{C}_s} = Y_u Y_+$  be the  $s$ -Iwasawa factorization of  $\tilde{Y}|_{\mathcal{C}_s}$  for any  $s \in (r, 1)$ . Since  $\tilde{Y}|_{\mathcal{C}_s}$  and  $Y_u$  are the boundaries of analytic maps on  $\mathcal{A}_s$  with nonzero determinants on  $\mathcal{A}_s \setminus \mathbb{S}^1$ , then  $Y_+$  is the boundary of an analytic map  $\tilde{Y}_+ : \mathcal{D}_1 \rightarrow \mathrm{GL}_2(\mathbb{C})$ . Then  $x_1 X = \tilde{Y}_+^* \tilde{Y}_+|_{\mathbb{S}^1}$ , so  $C = Y_+$  and  $f = x_1$  are the required maps.  $\square$

**4.2. Holomorphic vector bundles and unitarization.** We prove several pointwise and holomorphic lemmas relating to simultaneous unitarization.

**Lemma 4.4.** *Let*

$$L_\lambda : \mathbb{C}^m \rightarrow \mathbb{C}^n$$

*be a family of linear maps which depends analytically on  $\lambda \in \mathbb{C}^*$ . Let*

$$r = \min_{\lambda \in \mathbb{C}^*} \dim \ker L_\lambda.$$

*Then (i)  $\dim \ker L_\lambda = r$  on  $\mathbb{C}^* \setminus P$  for some subset  $P \in \mathbb{C}^*$  of isolated points, and (ii) there exists a trivial analytic rank- $r$  bundle  $E \rightarrow \mathbb{C}^*$  such that  $E_\lambda \subseteq \ker L_\lambda$  on  $\mathbb{C}^*$ , and  $E_\lambda = \ker L_\lambda$  on  $\mathbb{C}^* \setminus P$ .*

**Lemma 4.5.** *Let  $U_1, U_2 \in \mathrm{U}_2$  with  $[U_1, U_2] \neq 0$ . Let  $A \in \mathrm{GL}_2(\mathbb{C})$  and suppose that  $AU_1A^{-1} \in \mathrm{U}_2$  and  $AU_2A^{-1} \in \mathrm{U}_2$ . Then  $A \in \mathbb{R}^+ \times \mathrm{U}_2$ .*

*Proof.* Choose a basis for which  $U_1$  is diagonal. Factor  $A = UT$ , where  $U \in \mathrm{U}_2$  and  $T \in \mathcal{T}^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ . Then  $TU_1T^{-1} \in \mathrm{U}_2$  implies  $T$  is diagonal, and  $TU_2T^{-1} \in \mathrm{U}_2$  implies  $T \in \mathbb{R}^+ \mathrm{I}$ . Hence  $A = UT \in \mathbb{R}^+ \times \mathrm{U}_2$ .  $\square$

**Lemma 4.6.** (1) *Let  $M_1 \in \mathrm{GL}_2(\mathbb{C}) \setminus \{\pm \mathrm{I}\}$  be unitarizable. Let  $L_1 : \mathbb{M}_{2 \times 2}(\mathbb{C}) \rightarrow \mathbb{M}_{2 \times 2}(\mathbb{C})$  be the linear map defined by*

$$L_1(X) = XM_1 - M_1^{*-1}X.$$

*Then  $\dim \ker L_1 = 2$ .*

(2) *Let  $M_1, \dots, M_n \in \mathrm{GL}_2(\mathbb{C})$ ,  $n \geq 2$ , and suppose that  $\{M_1, \dots, M_n\}$  is simultaneously unitarizable and nondegenerate. Let  $L : \mathbb{M}_{2 \times 2}(\mathbb{C}) \rightarrow (\mathbb{M}_{2 \times 2}(\mathbb{C}))^n$  be the linear map defined by*

$$L(X) = (XM_1 - M_1^{*-1}X, \dots, XM_n - M_n^{*-1}X).$$

*Then  $\dim \ker L = 1$ .*

*Proof.* To show (1), by hypothesis there exists  $C \in \mathrm{GL}_2(\mathbb{C})$  such that  $CM_1C^{-1} \in \mathrm{SU}_2$ . Let  $X_0 = C^*C$ . A calculation shows that  $X_0 \in \ker L_1$  iff  $[X_0^{-1}X, M_1] = 0$ . Since the space of commutators with  $M$  is 2-dimensional, then  $\dim \ker L_1 = 2$  and  $\ker L_1 = \mathrm{span}\{X_0, X_0M_1\}$ .

To show (2), assume without loss of generality that  $M_1 \notin \{\pm I\}$ . By hypothesis there exists  $C \in \mathrm{GL}_2(\mathbb{C})$  such that  $CM_jC^{-1} \in \mathrm{SU}_2$ . Let  $X_0 = C^*C$ . Then  $X_0 \in \ker L$  so  $\dim \ker L \geq 1$ . But  $\ker L \subset \ker L_1$ , so  $\dim \ker L \leq 2$ .

Suppose  $\dim \ker L = 2$ . Then as above,  $\ker L = \mathrm{span}\{X_0, X_0M_j\}$  for each  $j$ . Hence for all  $i, j$ ,  $X_0M_i \in \mathrm{span}\{X_0, X_0M_j\}$ , so  $M_i \in \mathrm{span}\{I, M_j\}$  so  $[M_i, M_j] = 0$ , contrary to the hypothesis of the lemma.  $\square$

**Notation 4.7.** Let  $E \rightarrow \mathbb{S}^1$  be a vector bundle.  $E(\lambda)$  denotes the fiber of  $E$  over  $\lambda \in \mathbb{S}^1$ .  $E^*$  denotes the vector bundle whose fiber over  $\lambda \in \mathbb{S}^1$  is

$$\{\bar{X}^t \mid X \in E(\bar{\lambda}^{-1})\}.$$

**Lemma 4.8.** *Let  $E \rightarrow \mathbb{S}^1$  be a analytic line bundle such that (1)  $E^* = E$ , and (2) for each  $\lambda \in \mathbb{S}^1$  except possibly at finitely many points, there exists  $Y \in E(\lambda)$  which is positive definite. Then there exists a analytic section  $X$  of  $E$  such that  $X = X^*$ ,  $X$  is positive semidefinite on  $\mathbb{S}^1$ , and  $\det X \neq 0$ .*

*Proof.* Let  $X_1$  be a nowhere vanishing section of  $E$ . Then there exists  $\alpha \in \mathbb{C}^*$  such that  $X_2 = \alpha X_1 + (\alpha X_1)^* \neq 0$ , and  $X_2$  is a section of  $E$  satisfying  $X_2^* = X_2$ .

For any  $\lambda \in \mathbb{S}^1$  at which there exists  $Y \in E(\lambda)$  which is positive definite, since  $\dim E_\lambda = 1$  and  $Y \neq 0$ ,  $X_2(\lambda) = cY$  for some  $c \in \mathbb{C}$ . Since at  $\lambda$ ,  $X_2 = X_2^*$  and  $Y = Y^*$ ,  $c \in \mathbb{R}$ . Hence  $X_2(\lambda)$  is either positive definite, negative definite or 0 according as  $c > 0$ ,  $c < 0$  or  $c = 0$ .

Let  $P = \{p_1, \dots, p_n\} \subset \mathbb{S}^1$  be the set of points at which  $X_2$  switches between being positive and negative definite. Then  $P$  is even. Let  $f(\lambda) = \lambda^{-n} \prod_{j=1}^{2n} (\lambda - p_j)$ . Let  $p \in \mathbb{S}^1 \setminus P$  be a point for which  $X_2(p)$  is positive definite and let  $g(\lambda) = f(\lambda)/f(p)$ . Then  $g$  is analytic,  $g \neq 0$ ,  $g^* = g$ , and  $X_2$  is positive or negative definite according as  $g > 0$  or  $g < 0$ . Thus  $X = gX_2$  satisfies  $\det X \neq 0$  and  $X = X^*$  and is positive definite except at  $P$ , and is hence is positive semidefinite.  $\square$

**4.3. The gluing theorem.** We prove the main unitarization result: if a set of monodromies is unitarizable pointwise on  $\mathbb{S}^1$ , then it is unitarizable by an  $r$ -dressing. In the context of DPW, such a dressing closes the periods of the CMC immersion by lemma 1.7. The proof is based on lemmas 4.2–4.8.

**Theorem 4.9.** *Let  $M_k : \mathbb{S}^1 \rightarrow \mathrm{GL}_2(\mathbb{C})$  ( $k \in \{1, \dots, n\}$ ) be analytic maps such that the set  $\{M_1, \dots, M_n\}$  is nondegenerate and simultaneously unitarizable pointwise on  $\mathbb{S}^1$  except possibly at a finite subset of  $\mathbb{S}^1$ . Then there exists an analytic map  $C \in \Lambda_{\mp 1}^{+, \mathbb{R}} \mathrm{M}_{2 \times 2}(\mathbb{C})$  for which  $CM_k C^{-1}$  extends analytically across  $\{\det C = 0\}$  and is in  $\Lambda_1^* \mathrm{GL}_2(\mathbb{C})$ .*

*Moreover,  $C$  is unique up to multiplication by a scalar function  $\mathbb{S}^1 \rightarrow \mathbb{C}$  which is the boundary of an analytic function  $\mathcal{D}_1 \rightarrow \mathbb{C}^*$ .*

*Proof.* Let  $L_\lambda : \mathrm{M}_{2 \times 2}(\mathbb{C}) \rightarrow (\mathrm{M}_{2 \times 2}(\mathbb{C}))^n$  be the linear map defined by

$$L_\lambda(X) = (XM_1 - M_1^{*-1}X, \dots, XM_n - M_n^{*-1}X).$$

$L_\lambda$  depends analytically on  $\lambda \in \mathbb{C}^*$  because  $M_j$  do.  $L_\lambda$  is constructed so its kernel is the “square” of a unitarizer in the following sense: an analytic map  $C : \mathbb{S}^1 \rightarrow \mathrm{GL}_2(\mathbb{C})$  satisfies  $C^*C \in \ker L_\lambda$  if and only if  $CM_j C^{-1}$ ,  $j \in \{1, \dots, n\}$  satisfy the reality condition (1.2).

By lemma 4.6, for  $\lambda \in \mathbb{S}^1$  for which  $\{M_1, \dots, M_n\}$  is nondegenerate,  $\dim \ker L_\lambda = 1$ . By lemma 4.4(i), there exists a trivial analytic line bundle  $E \rightarrow \mathbb{S}^1$  such that  $E_\lambda = \ker L_\lambda$  except possibly at a finite subset of  $\mathbb{S}^1$ , where  $E_\lambda \subset \ker L_\lambda$ .  $E$  satisfies conditions (1) and (2) in the hypothesis of lemma 4.8, so by that theorem, there exists a analytic section  $X$  of  $E$  with the properties  $X = X^*$ ,  $X$  is positive semidefinite on  $\mathbb{S}^1$ , and  $\det X \neq 0$ .

By lemma 4.3, there exist a “square root” of  $X$  in the sense that there exist analytic maps  $C \in \Lambda_{\mp 1}^{+, \mathbb{R}} \mathrm{M}_{2 \times 2}(\mathbb{C})$  and  $f : \mathbb{S}^1 \rightarrow \mathbb{C}$  such that  $fX = C^*C$ . Then  $CM_j C^{-1}$  satisfies the conditions of lemma 2.3, so by that lemma it extends analytically across  $\{\det C = 0\}$  and is in  $\Lambda_1^* \mathrm{GL}_2(\mathbb{C})$ .

To show uniqueness, let  $C_1, C_2$  be two such maps, and let  $A = C_2 C_1^{-1}$ . Then  $A \in \Lambda_r^+ \mathrm{GL}_2(\mathbb{C})$  for every  $r \in (0, 1)$ . For each  $\lambda \in \mathbb{S}^1$  except possibly at a finite set  $S \subset \mathbb{S}^1$ ,  $A(\lambda)$  unitarizes the unitary matrices  $C_1 M_k C_1^{-1}|_\lambda$ . By lemma 4.5,  $A(\lambda) \in \mathbb{R}^+ \times \mathrm{U}_2$ . By lemma 2.3,  $A|_{\mathbb{S}^1} = fU$  for some meromorphic function  $f : \mathbb{S}^1 \rightarrow \mathbb{R}$  and analytic  $U \in \Lambda_1^* \mathrm{GL}_2(\mathbb{C})$ . For some  $r$  close to 1, the  $r$ -Iwasawa factorization of  $A|_{C_r}$  is then  $A|_{C_r} = U \cdot (fI)$ . But  $A|_{C_r} \in \Lambda_r^{+, \mathbb{R}} \mathrm{GL}_2(\mathbb{C})$ , so  $U = I$  and  $C_2 = fC_1$ .  $\square$

## 5. CONSTRUCTING TRINOIDS

Constructing trinoids is in the following steps:

1. Write down a family of DPW potentials on the thrice-punctured sphere which are locally gauge-equivalent to perturbations of the De-launay DPW potential at each puncture (definition 5.1).

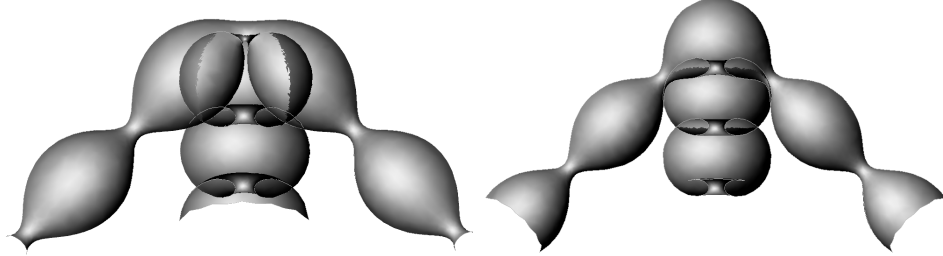


FIGURE 5. A pair of CMC trinoids with one nodoid end (neck-sizes  $(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6})$ ). The unduloid ends can be thought of as pulling outward along their axes, while the nodoid end pushes upward, in static equilibrium.

2. Show that the monodromy representation is unitarizable pointwise for  $\lambda \in \mathbb{S}^1$  (theorems 5.7 and 5.12).

3. Construct by the gluing theorem 4.9 a dressing for which the monodromy representation is unitary on  $\mathbb{S}^1$ . This dressing will close the three ends of the surface.

4. Show by the asymptotics theorem 3.4 that the three ends are asymptotically Delaunay.

**5.1. Trinoid potentials.** In this section a family of potentials is defined which will be used produce trinoids via the DPW construction. Near the punctures the potentials are local perturbations of Delaunay potentials via gauge equivalence. The family is parametrized by the three asymptotic Delaunay weights and has four connected components, divided according as the necksizes are positive or negative:  $[+++]$ ,  $[++-]$ ,  $[+--]$ ,  $[---]$ .

**Definition 5.1.** Let  $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ . Let  $w_1, w_2, w_3 \in (-\infty, 1) \setminus \{0\}$  and  $W = (w_1, w_2, w_3)$ . Let  $n_j = \frac{1}{2}(1 - \sqrt{1 - w_j})$ ,  $j = 1, 2, 3$  and suppose that  $n_j$  and  $w_j$  satisfy the inequalities

$$(5.1) \quad \begin{aligned} |n_1| + |n_2| + |n_3| &\leq 1 \\ |n_i| &\leq |n_j| + |n_k|, \quad \{i, j, k\} = \{1, 2, 3\} \end{aligned}$$

$$(5.2) \quad |w_i| \leq |w_j| + |w_k|, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Define  $\xi_W \in \Omega_{\Sigma}^1(\Lambda_1^{-1}\mathfrak{sl}_2(\mathbb{C}))$  by

$$(5.3) \quad \xi_W = \begin{pmatrix} 0 & \lambda^{-1}dz \\ (\lambda-1)^2Q_W/dz & 0 \end{pmatrix}$$

where

$$(5.4) \quad Q_W = \frac{w_3z^2 - (w_1 - w_2 + w_3)z + w_1}{16z^2(z-1)^2}dz^2$$

is the unique meromorphic quadratic differential on  $\mathbb{P}^1$  whose only poles are double poles at  $0, 1, \infty$  with respective quadratic residues  $w_k/16$ . By remark 1.1, the Hopf differential of the resulting CMC immersion will be  $-2H^{-1}Q_W\lambda^{-1}$ .

**5.2. Local gauge.** We show that the double pole of a trinoid potential can be gauged to a simple pole with Delaunay residue and, after a coordinate change, no constant term.

**Lemma 5.2.** *Let  $\xi_W \in \mathcal{T}$  be a trinoid potential. Then for each end  $p \in \{0, 1, \infty\}$  there exists a neighborhood  $U$  of  $p$ , an analytic map  $g : U^* \rightarrow \Lambda_1^+ \mathrm{GL}_2(\mathbb{C})$  and a conformal coordinate  $\tilde{z} : U \rightarrow \mathbb{C}$  with  $\tilde{z}(p) = 0$ , such that the expansion of  $\xi_W.g$  is*

$$(5.5) \quad \begin{pmatrix} 0 & a\lambda^{-1} + b \\ b + a\lambda & 0 \end{pmatrix} \frac{d\tilde{z}}{\tilde{z}} + O(\tilde{z})d\tilde{z}.$$

*Proof.* Let  $\mu_{w_1}$  as in equation (2.3). There exists  $a, b \in \mathbb{R}$  with  $|a| \geq |b|$  satisfying  $pp^* = \mu^2$ , where  $p = a\lambda^{-1} + b$ . Let

$$(5.6) \quad g_1 = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\lambda & \lambda p \end{pmatrix}.$$

Then  $\xi_W.g_1g_2$  has a simple pole at  $z = 0$  and residue as in equation (5.5). Let

$$g_3 = \mathbb{I} + \frac{k}{2} \begin{pmatrix} -1 & 0 \\ p^{-1} & 1 \end{pmatrix} z, \quad k = \frac{w_1 + w_2 - w_3}{2w_1}, \quad z = \tilde{z} - k\tilde{z}^2,$$

be the gauge and coordinate change constructed by lemma 3.2. Then  $g = g_1g_2g_3$  and  $\tilde{z}$  are the required gauge and coordinate change.  $\square$

**5.3. Gauge-equivalent trinoid potentials.** We present two gauge-equivalent forms of the trinoid potentials of definition 5.1.

**Lemma 5.3.**  *$\xi_W \in \mathcal{T}$  can be gauged globally to a potential which has simple poles with hermitian residues as in [10]. This gauge introduces extra poles with weight 0.*

*Proof.* We provide the gauge in the case of three positive weights. The proof in the other cases is similar. Let  $W = (w_1, w_2, w_3)$  and  $\xi_W \in \mathcal{T}$ . Let

$$\begin{aligned} w &= \frac{1}{2}(w_1 + w_2 + w_3), \\ r_k &= \frac{\sqrt{w - w_i} \sqrt{w - w_j}}{4\sqrt{w - w_k}}, \quad \{i, j, k\} = \{1, 2, 3\}, \\ r &= r_1 + r_2 + r_3, \\ p &= -\frac{1}{2r} + \sqrt{\frac{1}{4r^2} - 1}, \end{aligned}$$

taking positive square roots. Define the 1-forms  $\alpha, \beta, \gamma$  by

$$\begin{aligned} \alpha &= a dz = \left( \frac{r_1}{z} + \frac{r_2}{z-1} \right) dz \\ \beta &= b dz = \left( \frac{r-r_1}{z} + \frac{r-r_2}{z-1} - \frac{r}{z - \frac{r_1}{r_1+r_2}} \right) dz \\ \gamma &= \frac{1}{2}(p - p^{-1})(\alpha + \beta) \end{aligned}$$

Define the gauge  $g = g_1 g_2 g_3$ , where

$$\begin{aligned} g_1 &= \frac{1}{\sqrt{1+p^2\lambda}} \begin{pmatrix} 1 & & 0 \\ \frac{1}{2}\lambda(a'/a + (p+p^{-1})(a-b)) & 1 & \\ & & 1+p^2\lambda \end{pmatrix} \\ g_2 &= \begin{pmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{pmatrix} \\ g_3 &= \frac{1}{\sqrt{1-\lambda}} \begin{pmatrix} 1 & -p \\ -p^{-1}\lambda & 1 \end{pmatrix}. \end{aligned}$$

Then  $\xi_W$  gauged by  $g$  is

$$\xi_W \cdot g = \begin{pmatrix} \gamma & \alpha\lambda^{-1} + \beta \\ \beta + \alpha\lambda & -\gamma \end{pmatrix}.$$

The potential  $\xi_W \cdot g$  has simple poles at  $(0, 1, \infty, \frac{r_1}{r_1+r_2})$  with residues of the form (2.1) with respective weights  $(w_1, w_2, w_3, 0)$ .  $\square$

**Lemma 5.4.**  $\xi_W \in \mathcal{T}$  can be gauged to a potential of the form found in [3].

*Proof.* Let  $W = (w_1, w_2, w_3)$  and  $\xi_W \in \mathcal{T}$ . Let  $a_0, a_1 \in \mathbb{Z}$  and let  $\omega$  be an analytic loop on  $\mathbb{S}^1$  which extends to a holomorphic function on  $\mathcal{D}_1^*$  with no zeros, and extends meromorphically to 0 with  $\text{ord}_0 \omega = -1$ .

Define the gauge  $g = g_1 g_2$ , where

$$g_1 = \begin{pmatrix} (\lambda\omega)^{-1/2} z^{a_0/2} (z-1)^{a_1/2} & 0 \\ 0 & (\lambda\omega)^{1/2} z^{-a_0/2} (z-1)^{-a_1/2} \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\omega^{-1} z^{a_0} (z-1)^{a_1} (a_0/z + a_1/(z-1)) & 1 \end{pmatrix}$$

Then

$$\xi_W \cdot g = \begin{pmatrix} 0 & \sigma \\ \tau & 0 \end{pmatrix},$$

where

$$\sigma = \omega z^{-a_0} (z-1)^{-a_1}$$

$$-\sigma\tau = \frac{b_0}{z^2} + \frac{b_1}{(z-1)^2} + \frac{c}{z} - \frac{c}{z-1}$$

and

$$b_k = ((a_k - 1)/2)^2 - \mu_k^2, \quad k = 0, 1$$

$$c = 1/4 - a_0 a_1 / 2 - \mu_0^2 - \mu_1^2 + \mu_\infty^2$$

$$\mu_k = \frac{1}{2} \sqrt{1 + \frac{w_k (\lambda - 1)^2}{4\lambda}}, \quad k \in \{0, 1, \infty\}.$$

Then the potential  $\xi_W \cdot g$  is of the form found in [3].  $\square$

**5.4. Unitary monodromy on the thrice-punctured sphere.** In this section it is shown that given  $M_1, M_2, M_3 \in \text{SL}_2(\mathbb{C})$  whose product is I, the spherical triangle inequalities on the logs of their eigenvalues are necessary sufficient for the simultaneous unitarizability of  $M_1, M_2, M_3$ . An equivalent condition in terms of the traces of the matrices is given in [4]. Such inequalities are discussed in the context of holomorphic vector bundles in [1].

For a set of more than three matrices whose product is I, the spherical  $n$ -gon inequalities are necessary but not sufficient conditions for simultaneous unitarizability. The case  $n = 3$  is special in that the dimension of the set of conjugacy classes for  $M_1, M_2, M_3$  is the same as that of the eigenvalues.

**Lemma 5.5** (Spherical triangle inequalities). *Given  $(\nu_1, \nu_2, \nu_3) \in (0, \frac{1}{2})^3$ , there exists a nondegenerate spherical triangle on  $\mathbb{S}^1$  with sides  $2\pi\nu_k$  iff  $(\nu_1, \nu_2, \nu_3)$  satisfy the spherical triangle inequalities*

$$(5.7) \quad \begin{aligned} & \nu_1 + \nu_2 + \nu_3 < 1, \\ & \nu_i < \nu_j + \nu_k, \quad \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

**Lemma 5.6.** (i)  $M \in \mathrm{SL}_2(\mathbb{C})$  is unitarizable (notation 4.1) iff  $\frac{1}{2} \mathrm{tr} M \in (-1, 1)$  or  $M \in \{\pm I\}$ .

(ii) Any  $M \in \mathrm{SU}_2$  can be written  $M = \cos(2\pi\nu) + \sin(2\pi\nu)A$  with  $\nu \in [0, \frac{1}{2}]$  and  $A \in \mathfrak{su}_2$  with  $\det A = 1$ .

**Theorem 5.7.** Let  $M_1, M_2, M_3 \in \mathrm{SL}_2(\mathbb{C})$  with  $M_1 M_2 M_3 = I$  and with eigenvalues  $\exp(\pm 2\pi i \nu_k)$ ,  $\nu_k \in (0, \frac{1}{2})$ . Then  $M_1, M_2, M_3$  are nondegenerate and simultaneously unitarizable iff the spherical triangle inequalities (5.7) hold.

*Proof.* Suppose  $M_k$  are nondegenerate and simultaneously unitarizable, and let  $C$  be a unitarizer, so that  $CM_k C^{-1} \in \mathrm{SU}_2$ . Write  $CM_k C^{-1} = x_k I + y_k A_k$  as in lemma 5.6(ii). The nondegeneracy assumption means the  $A_k$  span  $\mathfrak{su}_2$ . Identifying  $\mathfrak{su}_2 \cong \mathbb{R}^3$ , let  $P_k$  be the planes perpendicular to  $A_k$  through 0. The planes intersect  $\mathbb{S}^2$  forming eight spherical triangles; consider one of the spherical triangle  $\Delta$  with side lengths less than  $\pi$ . A spherical trigonometry argument shows that the side lengths of  $\Delta$  are  $\nu_1, \nu_2, \nu_3$ , so by lemma 5.5, the spherical triangle inequalities (5.7) hold.

Conversely, given  $(\nu_1, \nu_2, \nu_3) \in (0, \frac{1}{2})^3$  satisfying the spherical triangle inequalities, by lemma 5.5 there exists a nondegenerate spherical triangle on  $\mathbb{S}^2$  with side lengths  $\nu_1, \nu_2, \nu_3$ . Let  $A_k$  be the normals to the planes through the sides. Then  $M_k = \cos(2\pi i \nu_k) I + \sin(2\pi i \nu_k) A_k$  are nondegenerate and unitary, and a spherical trigonometry argument shows that  $M_1 M_2 M_3 = I$ .

It remains to show that a choice  $(\nu_1, \nu_2, \nu_3)$  determines a unique conjugacy class of  $(M_1, M_2, M_3)$ . If  $(N_1, N_2, N_3)$  is another triple with the same traces, we can assume by conjugation that  $N_1 = M_1$ , and need to show that  $N_2$  is conjugate to  $M_2$  by a commutator of  $M_1$ . A computation shows

$$M_1^\circ M_2^\circ + M_2^\circ M_1^\circ = 2(t_3 - t_1 t_2) I,$$

where  $X^\circ$  denotes  $\mathrm{tracefree}(X)$ . Since  $\nu_1 \notin \{0, \frac{1}{2}\}$ , then  $M_1 \notin \{\pm I\}$  and  $M_1^\circ \neq 0$ . Fixing  $M_1^\circ$ , the equation is linear in  $M_2^\circ$  and has a 2 complex dimensional solution space. Since if  $M_2^\circ$  is a solution, so is  $CM_2^\circ C^{-1}$  for any commutator  $C$  of  $M_1$ , and the set of such commutators is also a 2 complex dimensional linear space, the set of solutions is a single orbit under conjugation by commutators of  $M_1$ . The result follows.  $\square$

**5.5. Unitarization of trinoid monodromy pointwise on  $\mathbb{S}^1$ .** We compute the eigenvalues of the monodromy for a potential  $\xi_W \in \mathcal{T}$ .

**Lemma 5.8.** Let  $\xi_W \in \mathcal{T}$  be a trinoid potential,  $\Phi$  a solution to the ODE  $d\Phi = \Phi \xi_W$ , and  $M_1, M_2, M_3$  the monodromy of  $\Phi$  at  $0, 1, \infty$

respectively. Then the eigenvalues of  $M_k$  are  $\exp(\pm 2\pi\nu_{w_k})$ , where  $\nu_w$  is defined by

$$(5.8) \quad \nu_w = \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{w(\lambda - 1)^2}{4\lambda}}.$$

*Proof.* By lemma 5.2,  $\xi_W$  is locally gauge-equivalent to a potential  $\eta$  of the form of equation (5.5). Let  $M_{\xi_W}$  and  $M_\eta$  be the respective monodromy representations of  $\xi_W$  and  $\eta$ . By lemma 3.3(i), the eigenvalues of the monodromy of  $\eta$  are  $\exp(\pm 2\pi i(\frac{1}{2} - \nu_w))$ , where  $\nu_w$  is given by equation (5.8). By lemma 1.10(ii),  $M_{\xi_W} = -M_\eta$ , hence the eigenvalues of the monodromy representation of  $\xi_W$  are  $\exp(\pm 2\pi i\nu_w)$ .  $\square$

Necessary and sufficient conditions are found that a monodromy representation with these eigenvalues be unitarizable for every  $\lambda \in \mathbb{S}^1$  (conditions (5.1)–(5.2)). The inequalities on the necks  $n_i$  are the spherical triangle inequalities on the eigenvalues evaluated at  $\lambda = -1$ . The inequalities on the weights  $w_i$  are implied by the balancing formula, according to which the sum of the end forces (the end axes in  $\text{su}_2$  with length  $w_k$ ) is 0.

**Notation 5.9.** Let  $T_0 \subset \mathbb{R}^3$  be the bounded set with tetrahedral boundary defined by

$$\begin{aligned} \nu_1 + \nu_2 + \nu_3 &\leq 1, \\ \nu_i &\leq \nu_j + \nu_k, \quad \{i, j, k\} = \{1, 2, 3\} \end{aligned}$$

and let  $T$  be the orbit of  $T_0$  by the action of the group generated by the transformations  $\nu_k \mapsto \nu_k + 1$  and  $\nu_k \mapsto -\nu_k$ .

**Lemma 5.10.** *Let  $\nu_{w_k}$  be defined by equation (5.8) and  $\nu = (\nu_1, \nu_2, \nu_3)$ . Then  $\nu \in T$  for all  $\lambda \in \mathbb{S}^1$  iff the inequalities (5.1) and (5.2) are satisfied.*

*Proof.* Assume equations (5.1) and (5.2) are satisfied. Define

$$\begin{aligned} \rho_k &= \frac{1}{2} - \frac{1}{2} \sqrt{1 - w_k x}, \quad \{i, j, k\} = \{1, 2, 3\} \\ f &= |\rho_1| + |\rho_2| + |\rho_3| \\ f_i &= -|\rho_i| + |\rho_j| + |\rho_k|, \quad \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

The terms in  $f$  are increasing, so  $f$  is increasing, so  $n_1 + n_2 + n_3 \leq 1$  implies that  $f \leq 1$  on  $[0, 1]$ . Hence  $\nu_1 + \nu_2 + \nu_3 \leq 1$  on  $\mathbb{S}^1$ .

In the case  $0 < w_1 \leq w_2$  or  $w_2 \leq w_1 < 0$ ,  $f_1$  is increasing, so  $n_1 \leq n_2 + n_3$  implies  $f_1$  is non-negative on  $[0, 1]$ . Hence  $\nu_1 \leq \nu_2 + \nu_3$  on  $\mathbb{S}^1$ .

We require the following fact: the function  $\rho_2/\rho_1$  extends to a  $C^\infty$  function at 0, and, if  $w_2 > w_1$ , then  $|\rho_2/\rho_1|$  is strictly increasing.

In the case  $w_1 \geq w_2$ ,  $w_1 \geq w_2$ , the above fact implies that that  $f_1/|\rho_1|$  is non-increasing.  $n_1 \leq n_2 + n_3$  implies that  $f_1/|\rho_1|$  is non-negative at 1, so  $f_1/|\rho_1|$ , and hence  $f_1$ , is non-negative on  $[0, 1]$ . Hence  $\nu_1 \leq \nu_2 + \nu_3$  on  $\mathbb{S}^1$ .

In the case  $w_1 \leq w_2$ ,  $w_1 \leq w_2$ , the above fact implies that that  $f_1/|\rho_1|$  is non-decreasing. But  $(f_1/|\rho_1|)(0) = -1 + |w_2/w_1| + |w_3/w_1| \geq 0$ , so  $f_1/|\rho_1|$  is non-negative on  $[0, 1]$ . Hence  $\nu_1 \leq \nu_2 + \nu_3$  on  $\mathbb{S}^1$ .

Symmetric arguments for the other cases imply that  $\nu \in T$ .

The proof of the converse is omitted.  $\square$

**Lemma 5.11.** *If the conditions (5.1)–(5.2) are satisfied, then  $w_k > -3$ ,  $k = 1, 2, 3$ .*

*Proof.* The inequalities  $|n_i| \leq |n_j| + |n_k| \leq 1 - |n_i|$  imply  $|n_i| \leq \frac{1}{2}$ . Hence  $w_i \geq -3$ . Suppose  $w_3 = -3$ , so  $n_3 = -\frac{1}{2}$ . By the above inequalities,  $|n_1| + |n_2| = \frac{1}{2}$ . Using  $w_r = 4n_r(1 - n_r)$ , the inequality  $|w_3| \leq |w_1| + |w_2|$  implies  $\frac{1}{4} \leq -n_1|n_1| - n_2|n_2|$ . An examination of cases according to the signs of  $n_1, n_2$  shows that this is satisfied only if  $n_1 = 0$  or  $n_2 = 0$ .  $\square$

The following theorem, the main theorem of the section, shows that the monodromy representation of a trinoid potential is pointwise unitarizable on  $\mathbb{S}^1$ .

**Theorem 5.12.** *Let  $\xi_W \in \mathcal{T}$ , and let  $\Phi$  a solution to the ODE  $d\Phi = \Phi\xi_W$  such that  $\Phi(p)$  is holomorphic on  $\mathcal{A}_0$  for some  $p$  in the universal cover of  $\Sigma$ . Then the monodromy representation of  $\Phi$  is nondegenerate and pointwise unitarizable on  $\mathbb{S}^1$  except possibly at finitely many points.*

*Conversely, the conditions (5.1)–(5.2) are necessary in order for the monodromy representation of  $\Phi$  to be nondegenerate and pointwise unitarizable on  $\mathbb{S}^1$  except possibly at finitely many points.*

*Proof.* By lemma 5.8, the eigenvalues of  $M_k$  are  $\exp(\pm 2\pi\nu_{w_k})$ , where  $\nu_{w_k}$  are defined by equation (5.8).

A necessary condition for the degeneracy of  $\{M_k\}$  on  $\mathbb{S}^1$  is  $\nu \in \partial T$ , but this occurs only at finitely many points on  $\mathbb{S}^1$ .

By the definition of  $\mathcal{T}$ ,  $w_1, w_2, w_3$  satisfy the neck and weight inequalities (5.1)–(5.2).

Let  $S = \{\lambda \in \mathbb{S}^1 \mid \nu \in \partial T\}$ . Then  $S$  is finite. Then the following are equivalent: (1)  $M_1, M_2, M_3$  are irreducible and simultaneously unitarizable on  $\mathbb{S}^1 \setminus S$ . (2)  $(\nu_1, \nu_2, \nu_3) \in T^\circ$  for all  $\lambda \in \mathbb{S}^1 \setminus S$  (theorem 5.7). (3)  $(\nu_1, \nu_2, \nu_3) \in T$  for all  $\lambda \in \mathbb{S}^1$ . (4) The inequalities (5.1) and (5.2) hold (lemma 5.10).  $\square$

### 5.6. Main theorem.

**Theorem 5.13.** *Let  $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ , let  $w_1, w_2, w_3 \in (-\infty, 1] \setminus \{0\}$  and  $n_j = \frac{1}{2}(1 - \sqrt{1 - w_j})$ ,  $j = 1, 2, 3$  and assume*

$$\begin{aligned} |n_1| + |n_2| + |n_3| &\leq 1 \\ |n_i| &\leq |n_j| + |n_k|, \quad \{i, j, k\} = \{1, 2, 3\} \\ |w_i| &\leq |w_j| + |w_k|, \quad \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

*Then there exists a conformal CMC immersion  $f : \Sigma \rightarrow \mathbb{R}^3$  with three ends which are asymptotic to half Delaunay surfaces with weights  $w_1, w_2, w_3$ .*

*Proof.* Let  $W = (w_1, w_2, w_3)$  and let  $\xi_W \in \mathcal{T}$  be a trinoid potential (definition 5.1). Let  $\tilde{\Sigma} \rightarrow \Sigma$  be the universal cover of  $\Sigma$  and  $\Gamma$  the group of deck transformations for this cover. Let  $\Phi \in \Lambda_1 \text{GL}_2(\mathbb{C})$  a nonsingular solution to the ODE  $d\Phi = \Phi \xi_W$  which extends analytically to  $\mathcal{A}_0$ . Let  $M_1, M_2, M_3$  the monodromies of  $\Phi$  at  $0, 1, \infty$  respectively.

*Step 1: Closing the ends.* By theorem 5.12, the set  $M_1, M_2, M_3$  is nondegenerate and pointwise simultaneously unitarizable on  $\mathbb{S}^1$  except possibly at a finite subset of  $\mathbb{S}^1$ . Thus by the gluing theorem 4.9 there exists an analytic map  $C \in \Lambda_{11}^{+, \mathbb{R}} \text{M}_{2 \times 2}(\mathbb{C})$  for which  $CM_k C^{-1}$  extends analytically across  $\{\det C = 0\}$  and is in  $\Lambda_1^* \text{GL}_2(\mathbb{C})$ . Let  $r = (0, 1)$  and  $f_\lambda = \text{Sym}_\lambda[\text{Uni}_r[C\Phi]]$ . By lemma 1.5,  $f$  is independent of the choice of  $r$ .

By lemma 3.3(ii), the eigenvalues of  $CM_k C^{-1}$  on  $\mathbb{S}^1$  are  $\exp(\pm 2\pi\nu_{w_k})$ . These by construction satisfy equation (1.11), so by lemma 1.8,  $CM_k C^{-1}$  satisfy the closing conditions (1.6). Hence by theorem 1.7,  $f_1$  is closed in the sense that  $\tau^* f_1 = f_1$  for all  $\tau \in \Gamma$ .

*Step 2: Delaunay asymptotics.* Choose an end  $p \in \{0, 1, \infty\}$ . By lemma 5.11, the corresponding weight  $w_k$  satisfies  $w_k > -3$ . By lemma 5.2, there exists a gauge  $g$  in a punctured neighborhood of  $p$  such that after a coordinate change,  $\xi_W \cdot g$  has no constant term in its series expansion. Since  $M_{C\Phi} \in \Lambda_r^* \text{GL}_2(\mathbb{C})$  by the construction of  $C$ , and  $M_{C\Phi} = -M_{C\Phi g}$  by lemma 1.10(ii), then  $M_{C\Phi g} \in \Lambda_r^* \text{GL}_2(\mathbb{C})$ . Hence by the asymptotics theorem 3.4,  $\text{Sym}_1[\text{Uni}_r[C\Phi g]]$  is asymptotic to half Delaunay surfaces at its ends  $(0, 1, \infty)$  with respective weights  $w_1, w_2, w_3$ . By lemma 1.10(iii) the same is true for  $f_1$ .  $\square$

The following theorem discusses the symmetry groups of the trinoids constructed in theorem 5.13.

**Theorem 5.14.** *(i) Each trinoid in the family constructed in theorem 5.13 has a plane of reflective symmetry which fixes each end. (ii) Each*

*isosceles trinoid in the family has a further plane of reflective symmetry perpendicular to this plane which exchanges the equal ends and fixes the third end. (iii) Each equilateral trinoid in the family has the order-12 symmetry group of an equilateral triangle slab.*

The proof of this theorem will be found in [11], which discusses gauge symmetries in the general context of  $n$ -noids.

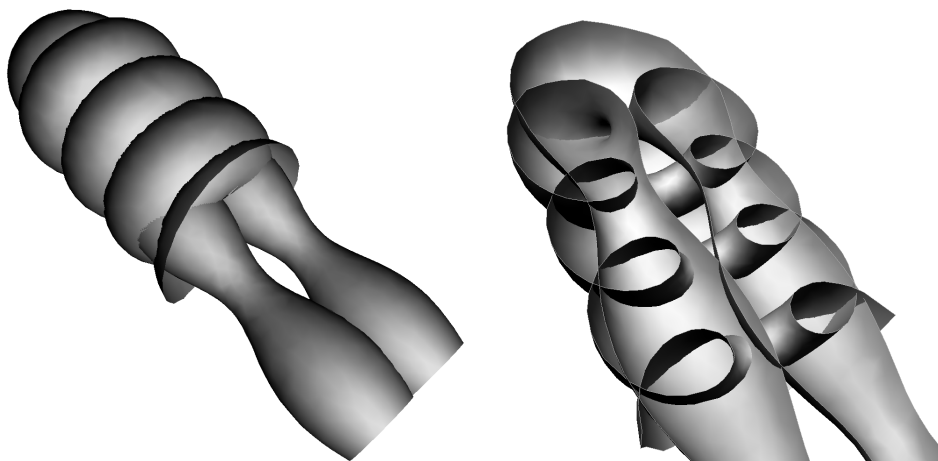


FIGURE 6. Two views of a trinoid with one nodoid end (necksizes  $(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$ ). The weight of the nodoid end is twice that of each unduloid end but opposite in sign, so the end axes are parallel as required by balancing.

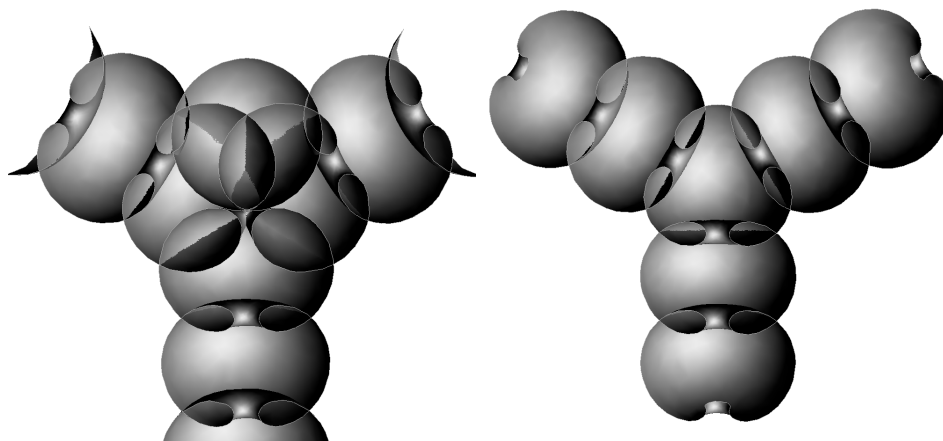


FIGURE 7. A pair of CMC trinoids with three nodoid ends (necksizes  $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$ ).

## 6. OPEN QUESTIONS

1. Computer experiments indicate that the trinoids in the subfamily with embedded ends are Alexandrov embedded.
2. Bäckland transformations can be applied to Delaunay surfaces to obtain bubbletons [12]. Construct Bäckland transformations of CMC  $n$ -noids.
3. Classify the CMC trinoids.
4. Construct and classify the CMC  $n$ -noids. This will involve unitarizing the monodromy representation on the  $n$ -punctured sphere.
5. Construct and classify  $n$ -noids with genus  $> 0$ .

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