CONSTANT MEAN CURVATURE n-NOIDS WITH SYMMETRIES

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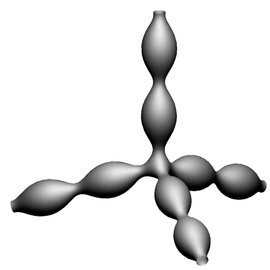
Abstract. In this note I construct two families of constant mean curvature spheres with Delaunay ends via DPW. The construction uses symmetries of the DPW representation to unitarize the holonomies of the DPW potential.



Introduction

In this note I construct two families of constant mean curvature spheres with Delaunay ends (noids) via DPW. The first family ("flowers") consists of noids with n coplanar end axes and the symmetry of a n-gon slab. The members of the second family ("pods") have n non-coplanar end axes and the symmetry of a $(n-1)$-gon. The special case of the family of pods with four equal-weight ends have the symmetry of a tetrahedron.

The construction takes advantage of symmetries of the surface to unitarize the holonomies of the DPW potential. The symmetry of the potential induces a gauge symmetry via a loop g_0 on the end mondromies M_j:

$$M_j = g_0^{-j} M_0 g_0^j.$$

M_0 and g_0 are shown to be simultaneously unitarizable on $\mathbb{S}^1$. and so the M_j are simultaneously unitarizable. It follows that there exists a dressing which closes the ends of the surface.

This research was supported by National Science Foundation grant DMS-00-76085.

2000 *Mathematics Subject Classification.* 53A10.

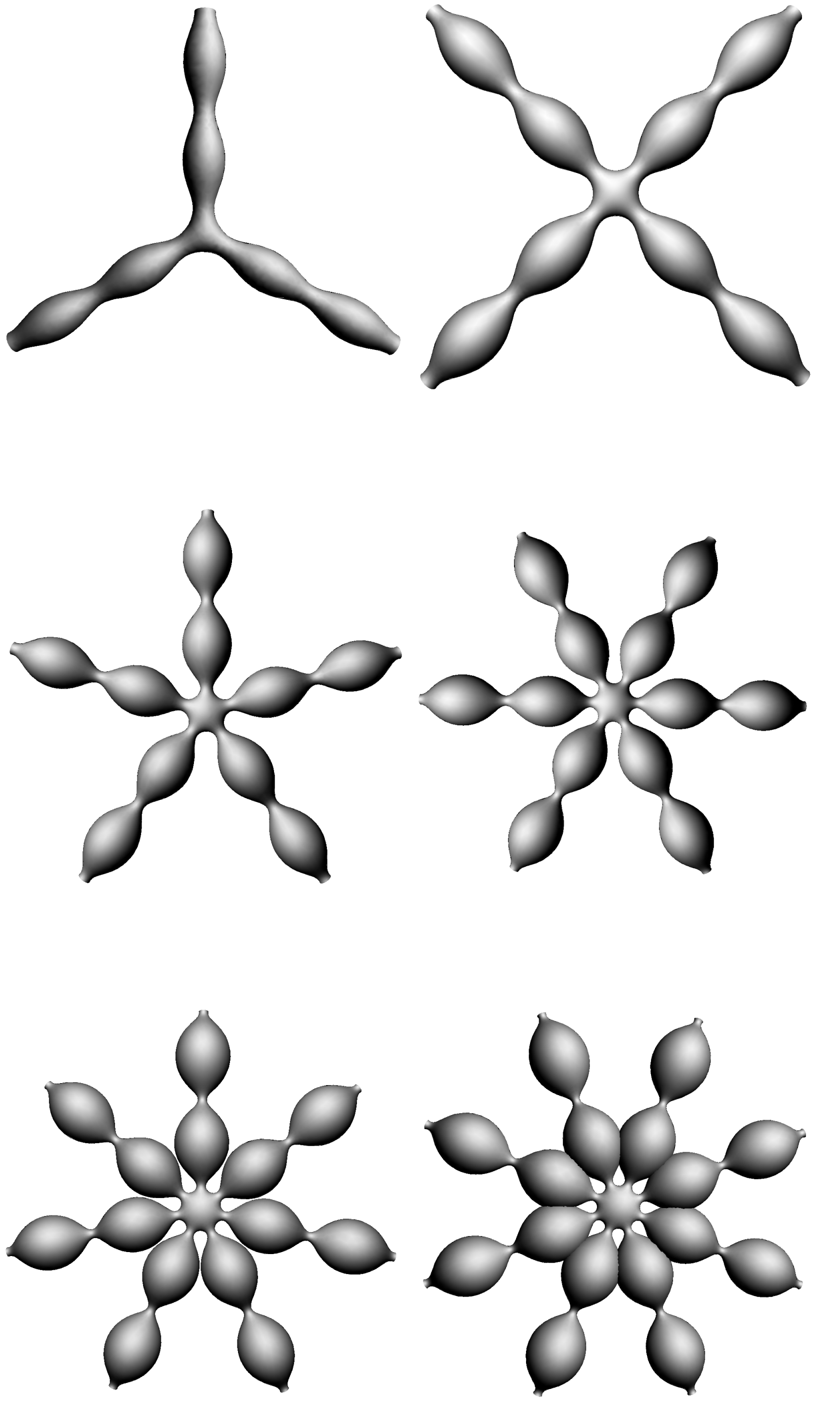


FIGURE 1. CMC n-flowers: n-noids with coplanar end axes, equal necksizes, and generically, a symmetry group of order $4n$.

1. Symmetries

Lemmas 1.3 and 1.4 are basic results showing how a gauge symmetry of a potential descends to the ODE solution, its monodromy, the extended frame and the CMC immersion. The two lemmas detail the orientation-preserving and orientation-reversing cases respectively.

Notation 1.1. Let Σ be a Riemann surface. For a DPW potential $\xi \in \Omega^1_1(\Lambda^{-1}_\Sigma \mathrm{sl}_2(\mathbb{C}))$, and a map $g : \Sigma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$, the *gauged potential* is

$$\xi{\cdot}g = g^{-1}dg + g^{-1}\xi g.$$

Gauging has the properties:

$$d\Phi = \Phi\xi \implies d(\Phi g) = (\Phi g)(\xi{\cdot}g)$$

and

$$(\xi{\cdot}g_1)\cdot g_2 = \xi{\cdot}(g_1 g_2).$$

Notation 1.2. Let Σ be a Riemann surface, $\widetilde{\Sigma}$ its universal cover, and Γ its group of deck transformations. Let $z_0 \in \widetilde{\Sigma}$. Let $\xi \in \Omega^1_1(\Lambda^{-1}_\Sigma \mathrm{sl}_2(\mathbb{C}))$. Let $\Phi : \widetilde{\Sigma} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be a solution to ODE $d\Phi = \Phi\xi$. The monodromy representation for Φ relative to the basepoint z_0 is the map $M : \Gamma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ defined by $M_\sigma = \Phi(\sigma(z_0))\Phi(z_0)^{-1}$.

Lemma 1.3. *Let Σ be a Riemann surface, $\pi : \widetilde{\Sigma} \to \Sigma$ its universal cover, and Γ its group of deck transformations. Let $z_0 \in \widetilde{\Sigma}$. Let $\xi \in \Omega^1_1(\Lambda^{-1}_\Sigma \mathrm{sl}_2(\mathbb{C}))$ and let $M : \Gamma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be the monodromy representation for Φ relative to the basepoint z_0.*

Let $\widetilde{\tau} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ be a lift of an orientation-preserving automorphism $\tau : \Sigma \to \Sigma$. Let $g : \Sigma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be a map such that

$$\tau^*\xi = \xi{\cdot}g. \tag{1.1}$$

and let $V = ((\widetilde{\tau}^*\Phi)(z_0))g^{-1}(z_0)\Phi^{-1}(z_0)$.

Then (1)

$$\widetilde{\tau}^*\Phi = V\Phi g, \tag{1.2}$$

and (2)

$$\widetilde{\tau}^*M = \mathrm{Ad}_V\, M. \tag{1.3}$$

Assume $V \in \Lambda^*_r\mathrm{SL}_2(\mathbb{C})$. Then (3) with $F = \mathsf{Uni}_r(\Phi)$,

$$\widetilde{\tau}^*F = VFD \tag{1.4}$$

for some λ-independent diagonal matrix D, and (4) with $f_\lambda = \mathsf{Sym}_\lambda[F]$.

$$\widetilde{\tau}^*f_\lambda = \mathrm{Ad}_V\, f_\lambda - \tfrac{2}{H}V'V^{-1}, \tag{1.5}$$

so τ^*f_λ is related to f_λ by an orientation-preserving isometry of su_2.

Proof. The gauge symmetry (1.1) implies equation (1.2), since $\widetilde{\tau}^*\Phi$ and $V\Phi g$ each satisfies the equation

$$\Psi^{-1}d\Psi = \widetilde{\tau}^*\xi = \xi\cdot g$$

and they have the same value at z_0. Equation (1.3) follows.

Now assume $V \in \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$. Let $\Phi = FB$ be the Iwasawa factorization of Φ. Then the Iwasawa factorization of $\widetilde{\tau}^*\Phi$ is $(\widetilde{\tau}^*F)(\widetilde{\tau}^*B)$, while that of $V\Phi g$ is $(VFD)(D^{-1}Bg)$ for some λ-independent diagonal matrix D. Equating unitary parts yields equation (1.4). The Sym formula yields equation (1.5). □

Lemma 1.4. *Let Σ be a Riemann surface, $\pi : \widetilde{\Sigma} \to \Sigma$ its universal cover, and Γ its group of deck transformations. Let $z_0 \in \widetilde{\Sigma}$. Let $\xi \in \Omega_1^1(\Lambda_\Sigma^{-1}\mathrm{sl}_2(\mathbb{C}))$ and let $M : \Gamma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be the monodromy representation for Φ relative to the basepoint z_0.*

Let ι be an orientation-reversing involution of the λ-plane. Let $\widetilde{\tau} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ be a lift of an orientation-preserving automorphism $\tau : \Sigma \to \Sigma$. Let $g : \Sigma \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ a map such that

$$\overline{\tau^*\xi(\iota\lambda)} = \xi\cdot g. \tag{1.6}$$

Let $V(\lambda) = \overline{(\widetilde{\tau}^*\Phi(\iota\lambda))(z_0)}g^{-1}(z_0,\, \lambda)\Phi^{-1}(z_0,\, \lambda)^{-1}$.

Then (1)

$$\overline{\widetilde{\tau}^*\Phi(\iota\lambda)} = V\Phi g, \tag{1.7}$$

and (2)

$$\overline{\widetilde{\tau}^* M(\iota\lambda)} = \mathrm{Ad}_V\, M(\lambda). \tag{1.8}$$

Assume $V \in \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ and $\iota : \lambda \mapsto \overline{\lambda}$. Then (3) with $F = \mathsf{Uni}_r(\Phi)$,

$$\overline{\widetilde{\tau}^* F(\overline{\lambda})} = VFD \tag{1.9}$$

for some λ-independent diagonal matrix D, and (4) with $f_\lambda = \mathsf{Sym}_\lambda[F]$.

$$\overline{\widetilde{\tau}^* f_{\overline{\lambda}}} = -\big(\mathrm{Ad}_V\, f_\lambda - \tfrac{2}{H}V'V^{-1}\big). \tag{1.10}$$

so $\widetilde{\tau}^* f_1$ is related to f_1 by an orientation-reversing isometry of su_2.

Proof. The gauge symmetry (1.6) implies equation (1.7), because $\overline{\widetilde{\tau}^*\Phi(\iota\lambda)}$ and $V\Phi g$ each satisfies the equation

$$\Psi^{-1}d\Psi = \overline{\tau^*\xi(\iota\lambda)} = \xi\cdot g$$

and they have the same value at z_0. Equation (1.8) follows.

Now assume $V \in \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ and $\iota : \lambda \mapsto \overline{\lambda}$. Let $\Phi = FB$ be the Iwasawa factorization of Φ. Then the Iwasawa factorization of $\overline{\widetilde{\tau}^*\Phi(\overline{\lambda})}$ is

$$\overline{\widetilde{\tau}^*\Phi(\overline{\lambda})} = (\overline{\widetilde{\tau}^*F(\overline{\lambda})})(\overline{\widetilde{\tau}^*B(\overline{\lambda})}),$$

while that of $U\Phi g$ is $(UFD)(D^{-1}Bg)$ for some λ-independent diagonal matrix D. Equating unitary parts yields equation (1.9). The Sym formula yields equation (1.10). □

Remark 1.1. A useful case of the above lemma is when $\widetilde{\tau} \circ \sigma = \sigma^{-1}$, so $\widetilde{\tau}^* M_\sigma = M_{\widetilde{\tau}\circ\sigma} = {M_\sigma}^{-1}$.

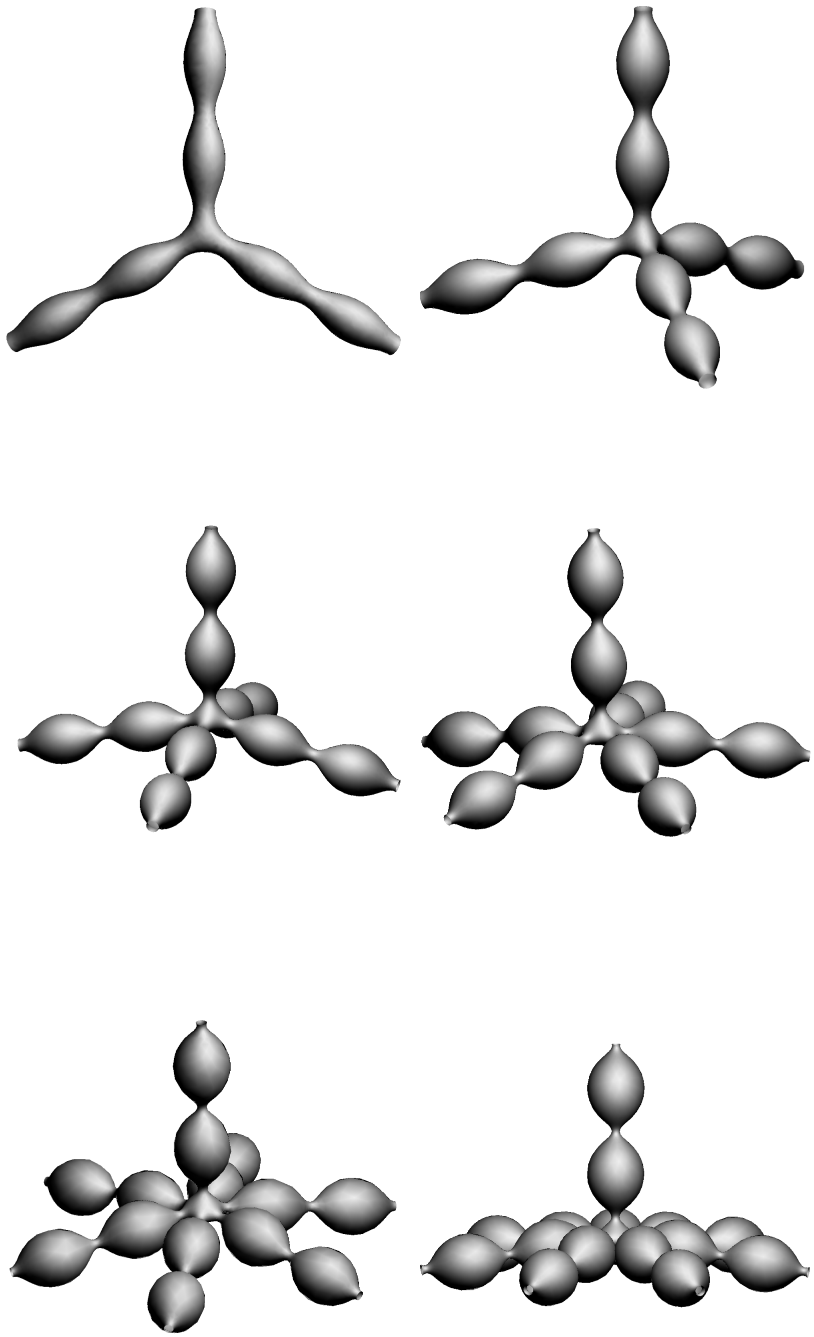


FIGURE 2. CMC n-pods: n-noids which generically have noncoplanar end axes, equal necksizes at $n-1$ ends, and a dihedral symmetry group of order $2(n-1)$.

2. K-NOIDS

We now construct two families of n-noids. The first family contains noids with n equal-weight ends with coplanar axes. The noids in the second family have $n-1$ equal-weight ends, with generically noncoplanar end axes.

First we defined the DPW potential.

Definition 2.1. Let $m \geq 1$ and let $w_1 \in (-\infty, 1] \setminus \{0\}$, $w_\infty \in (-\infty, 1]$ and $v_k = \frac{1}{2}(1 - \sqrt{1 - w_k})$, $k \in \{1, \infty\}$ satisfy

$$\begin{aligned} &0 \leq m|\nu_1| + \nu_\infty \leq 1 \\ &0 \leq m|\nu_1| - \nu_\infty \leq m-1 \\ &|w_\infty| \leq m|w_1|. \end{aligned} \tag{2.1}$$

Let

$$\Sigma = \begin{cases} \mathbb{P}^1 \setminus \left(\{z^m = 1\} \cup \{\infty\}\right) & w_\infty \neq 0 \\ \mathbb{P}^1 \setminus \{z^m = 1\} & w_\infty = 0 \end{cases} \tag{2.2}$$

with standard conformal coordinate z. Define $\xi_{m,w_1,w_\infty} \in \Omega^1_1(\Lambda^{-1}_\Sigma \mathrm{sl}_2(\mathbb{C}))$ by

$$\xi_{m,w_1,w_\infty} = \begin{pmatrix} 0 & \lambda^{-1} \\ \dfrac{z^{m-2}(m^2 w_1 + w_\infty(z^m - 1))}{16(z^m - 1)^2}(1-\lambda)^2 & 0 \end{pmatrix} dz \tag{2.3}$$

Theorem 2.2. *Let ξ_{n,w_1,w_∞} as in definition 2.1. Then there exists a dressing matrix C such that the CMC immersion constructed by DPW from ξ_{m,w_1,w_∞}, dressed by C, is a CMC noid.*

In the case $w_\infty \neq 0$, the resulting CMC immersion has $m+1$ Delaunay ends: m weight w_1 ends at the m'th roots of unity and one weight w_∞ end at ∞. The surface has an order $m-2$ umbilic point at 0 and m order 1 umbilic points at the m'th roots of $1 - m^2 w_1/w_\infty$.

In the case $w_\infty = 0$, the resulting CMC immersion has m weight w_1 Delaunay ends at the m'th roots of unity. The potential $\xi_{m,w_1,0}$ has an additional double pole at ∞ which is a smooth point of the resulting CMC immersion. The surface has two order $m-1$ umbilics points at 0 and ∞.

Proof. The potential $\xi = \xi_{m,w_1,w_\infty}$ has the following symmetry:

$$\rho : z \longmapsto e^{2\pi i/m} z, \quad g = \begin{pmatrix} e^{-\pi i/m} & 0 \\ 0 & e^{\pi i/m} \end{pmatrix}, \quad \rho^* \xi = \xi \cdot g. \tag{2.4}$$

By lemma 1.3, the mondromies have the symmetry

$$M_j = g_0^{-j} M_0 g_0^j.$$

Step 1: closing the ends. We have that

$$\tfrac{1}{2}\operatorname{tr} g_0^{-1} = \cos \pi/m \tag{2.5}$$

$$\tfrac{1}{2}\operatorname{tr} M_0 = \cos 2\pi\nu_1 \tag{2.6}$$

$$\tfrac{1}{2}\operatorname{tr} g^{-1}M_0 = \cos 2\pi(\tfrac{1}{2}-\nu_\infty)/m. \tag{2.7}$$

[NOTE: know third trace is root of unity. Fix which one by evaluation at $\lambda = 1$.] Hence g^{-1}, M_0 are simultaneously unitarizable on $\mathbb{S}^1 \setminus \{\pm 1\}$ iff the spherical triangle inequalities hold for $(1/(2m)$, ν_1, $(\frac{1}{2}-\nu_\infty)/m)$ on $\mathbb{S}^1 \setminus \{\pm 1\}$. A calculus argument shows that these are equivalent to inequalities (2.1).

By the gluing theorem [1], there exists an r-dressing C for r close to 1 for which the resulting immersion is closed at all its ends. □

Corollary 2.3. *Let ξ_{n,w_1,w_∞} as in definition 2.1.*

In the case $w_\infty \neq 0$, (1) for $m > 2$ the end axes are non-coplanar; (2) (i) In the case that $m > 3$, or the case $m = 3$ where not all necksizes are equal, the group of ambient isometries of the surface in $\mathbb{R}^3$ is the dihedral group of a regular m-gon, of order $2m$; (ii) In case of $m = 3$ and equal necksizes, the ambient isometry group is that of the Platonic tetrahedron, of order 24.

In the case $w_\infty \neq 0$, (1) the end axes lie in a plane; (2) For $m \geq 3$, the group of ambient isometries of the surface in $\mathbb{R}^3$ is that of a regular m-gon slab, with order $4m$.

Proof. Using the symmetry of equation (2.4) and the fact that $g_0^n = -\mathrm{I}$, lemma 1.3 implies that the surface has a orientation-preserving isometry induced by ρ of order n, which is hence a rotation.

The potential $\xi = \xi_{m,w_1,w_\infty}$ has the following further symmetry:

$$\sigma : z \longmapsto \overline{z}, \quad \overline{\sigma^*\xi(\overline{\lambda})} = \xi. \tag{2.8}$$

By lemma 1.4, the surface has an orientation-reversing isometry induced by σ of order 2 each, which is hence a reflection in a plane.

Now assume the special case $n = 4$, $w_1 = w_\infty$, of four equal-weight ends. We show that the extrinsic symmetry group of the resulting surface is that of the regular tetrahedron. In this case, $\xi = \xi_{4,w_1,w_1}$ has the additional symmetry:

$$\upsilon : z \longmapsto \frac{z+2}{z-1}, \quad k = i\sqrt{3}\begin{pmatrix} z-1 & 0 \\ -\lambda & -3/(z-1)\end{pmatrix}, \quad \xi{\cdot}k = \upsilon^*\xi. \tag{2.9}$$

By lemma 1.3, the resulting CMC immersion has a order-2 rotation which exchanges two ends and fixes the other two.

Now take the case $w_\infty = 0$. $\xi_{n,w_1,0}$ has the additional symmetry

$$\tau : z \longmapsto 1/\overline{z}, \quad h = i\begin{pmatrix} z & 0 \\ -1 & -z^{-1} \end{pmatrix}, \quad \overline{\tau^*\xi(\overline{\lambda})} = \xi{\cdot}h. \tag{2.10}$$

By lemma 1.4, the surface has an orientation-reversing isometry induced by τ of order 2, which is hence a reflection in a plane. □

References

1. N. Schmitt, *Constant mean curvature trinoids*, in preperation, 2002.

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